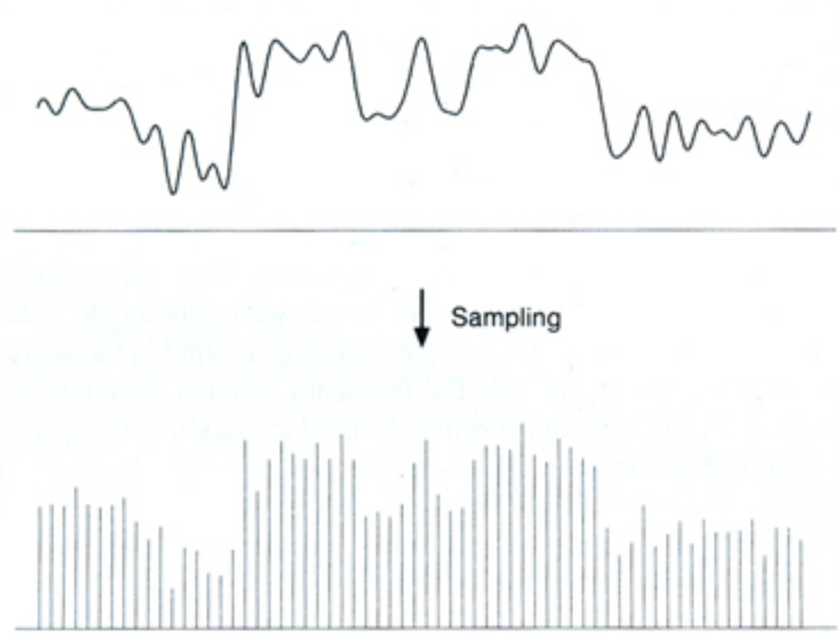
Sampling and reconstruction

CS 465 Lecture 5

Sampled representations

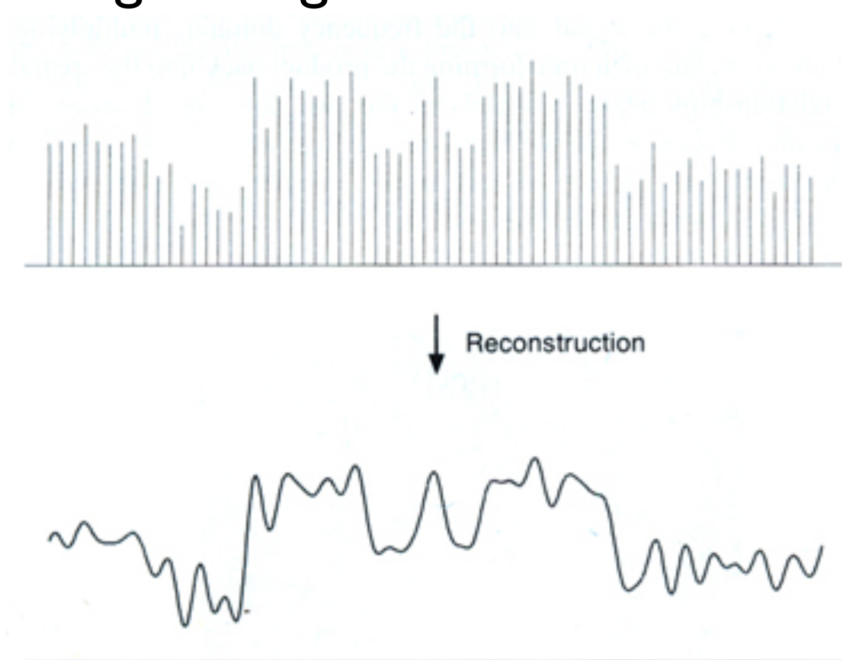
- How to store and compute with continuous functions?
- Common scheme for representation: samples
  - write down the function's values at many points



[FvDFH fig.14.14b / Wolberg]

Reconstruction

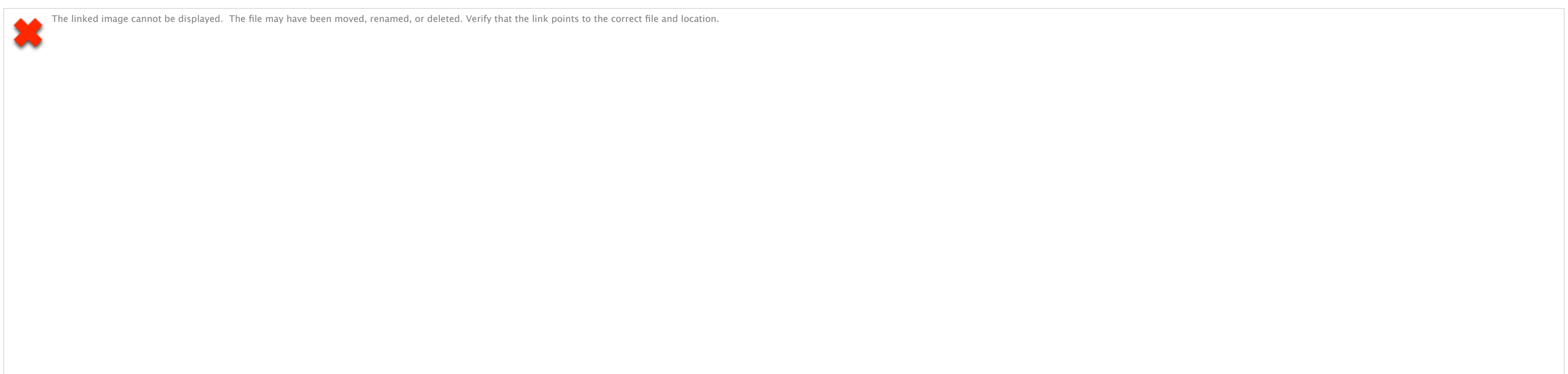
- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to "guessing" what the function did in between



[FvDFH fig.14.14b / Wolberg]

Filtering

- Processing done on a function
  - can be executed in continuous form (e.g. analog circuit)
  - but can also be executed using sampled representation
- Simple example: smoothing by averaging



Roots of sampling

- Nyquist 1928; Shannon 1949
  - famous results in information theory
- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc
  - the first high-profile consumer application
- This is why all the terminology has a communications or audio "flavor"
  - early applications are 1D; for us 2D (images) is important

Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  - how can we be sure we are filling in the gaps correctly?

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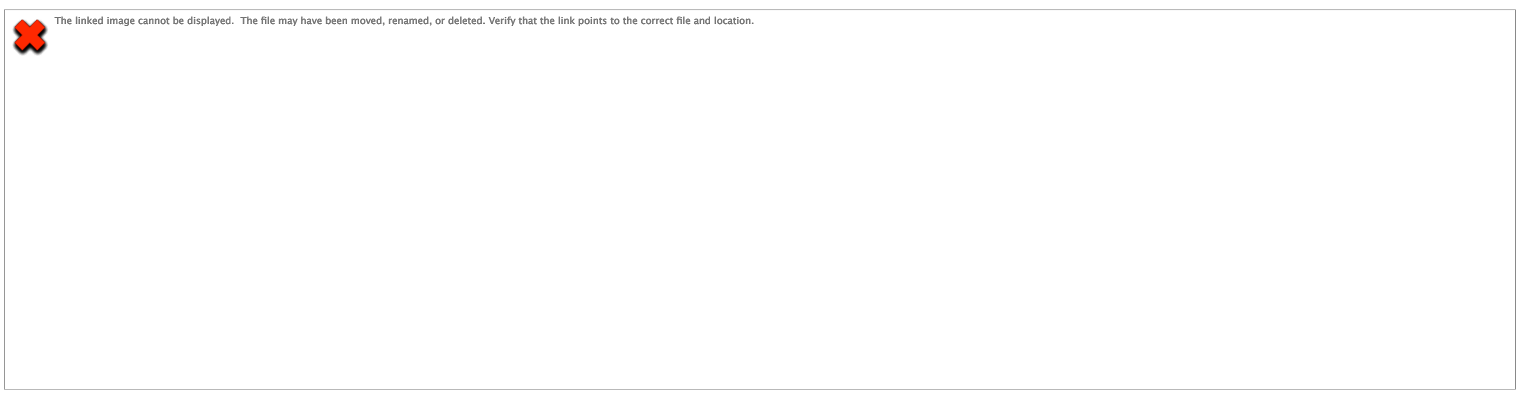
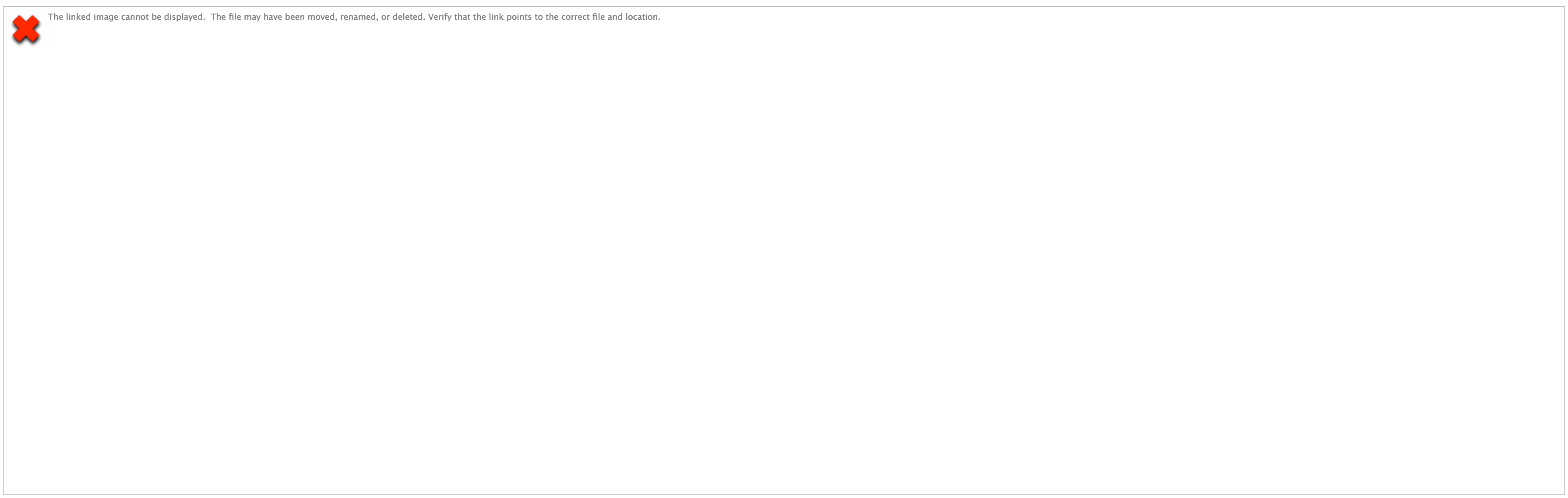
Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)

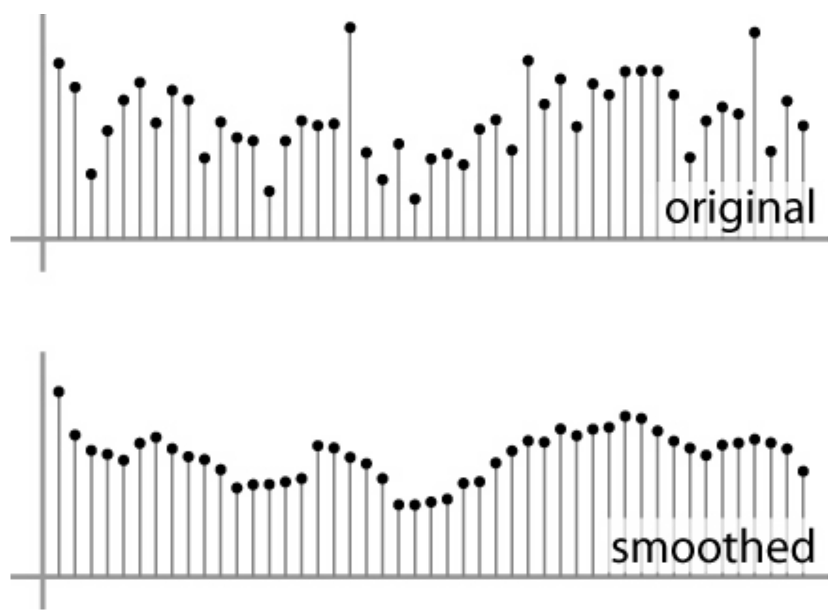


Linear filtering: a key idea

- Transformations on signals; e.g.:
  - bass/treble controls on stereo
  - blurring/sharpening operations in image editing
  - smoothing/noise reduction in tracking
- Key properties
  - linearity: filter(f + g) = filter(f) + filter(g)
  - shift invariance: behavior invariant to shifting the input
    - delaying an audio signal
    - sliding an image around
- Can be modeled mathematically by *convolution*

Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



Convolution warm-up

- Same moving average operation, expressed mathematically:

$$b[k] = \frac{1}{2r+1} \sum_{i=k-r}^{k+r} a[k]$$

Discrete convolution

- Simple averaging:

$$b[k] = \frac{1}{2r+1} \sum_{i=k-r}^{k+r} a[k]$$

  - every sample gets the same weight
- Convolution: same idea but with *weighted* average

$$b[k] = \sum_i c[i] a[k-i]$$

  - each sample gets its own weight (normally zero far away)
- Sequence of weights c_i is called a *filter*
  - support, symmetry

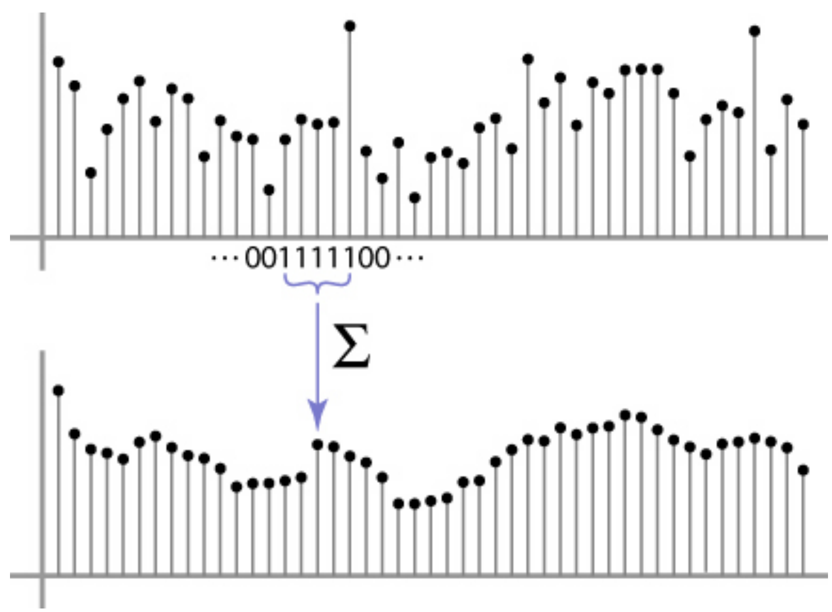
Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
  - commutative $a \star b = b \star a$
  - associative $a \star (b \star c) = (a \star b) \star c$
  - distributes over addition $a \star (b + c) = a \star b + a \star c$
  - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
  - identity: unit impulse e = […, 0, 0, 1, 0, 0, …]

    $a \star e = a$
- Conceptually no distinction between filter and signal

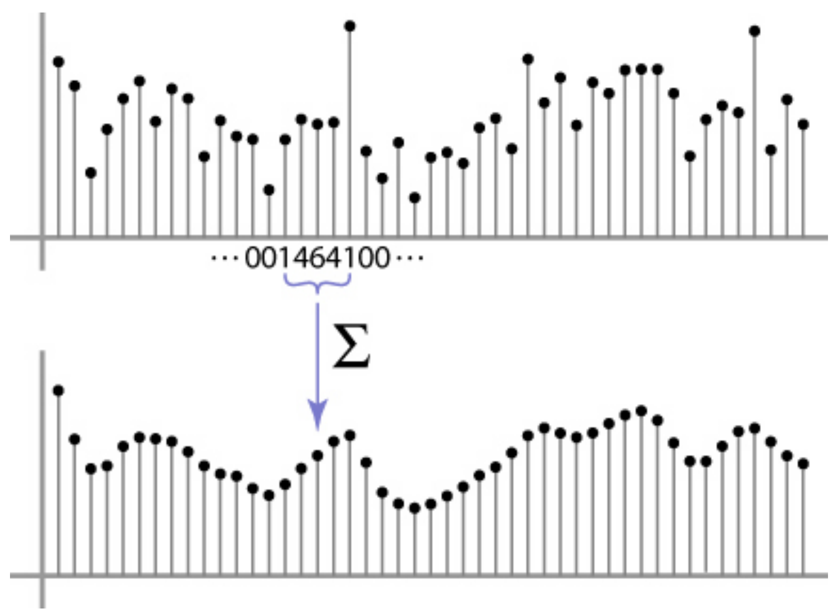
Convolution and filtering

- Can express sliding average as convolution with a *box filter*
- $c_{box} = [\ldots, 0, 1, 1, 1, 1, 1, 0, \ldots]$



Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) […, 1, 4, 6, 4, 1, …]



Discrete filtering in 2D

- Same equation, one more index

$$b[k,l] = \sum_{i,j} c[i,j]a[k-i,l-j]$$

  - now the filter is a rectangle you slide around over a grid of numbers
- Commonly applied to images
  - blurring (using box, using gaussian, …)
  - sharpening (impulse minus blur)
  - usefulness of associativity


[Philip Greenspun]
original ▲ | ▼ box blur
sharpened ▲ | ▼ gaussian blur

Optimization: separable filters

- basic alg. is $O(r^2)$: large filters get expensive fast!
- definition: $h(x,y)$ is *separable* if it can be written as:

$$h[x,y] = h_x[x]h_y[y]$$

  – this is a useful property for filters because it allows factoring:

$$\begin{aligned} g[x,y] &= \sum_i \sum_j h[i,j] f[x-i, y-j] \\ &= \sum_i \sum_j h_x[i] h_y[j] f[x-i, y-j] \\ &= \sum_i h_x[i] \left(\sum_j h_y[j] f[x-i, y-j] \right) \end{aligned}$$

Separable filtering

$$h[x,y] = h_x[x]h_y[y]$$

1	4	6	4	1
4	16	24	16	4
6	24	36	24	6
4	16	24	16	4
1	4	6	4	1

0	0	0	0	0
0	0	0	0	0
1	4	6	4	1
0	0	0	0	0
0	0	0	0	0

0	0	1	0	0
0	0	4	0	0
0	0	6	0	0
0	0	4	0	0
0	0	1	0	0

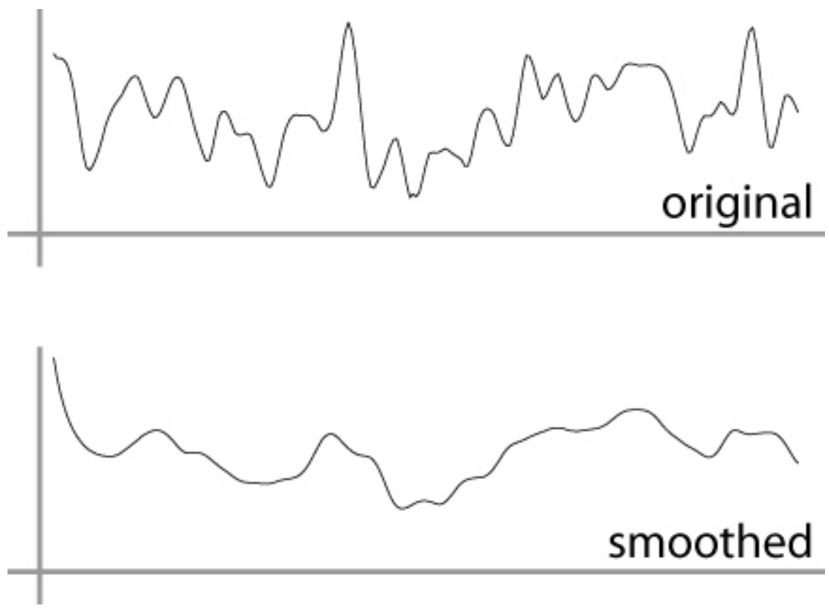
second, convolve with this

first, convolve with this

$$\sum_i h_x[i] \left(\sum_j h_y[j] f[x-i, y-j] \right)$$

Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well
  - output is continuous
  - integration replaces summation



Continuous convolution

- Sliding average expressed mathematically:

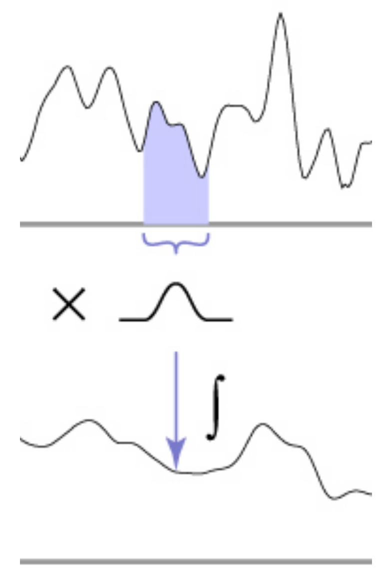
$$g(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(t)dt$$

  – note difference in normalization (only for box)

- Convolution just adds weights

$$g(x) = \int_{-\infty}^{\infty} h(t) f(x-t) dt$$

  – weighting is now by a function
  – weighted integral is like weighted average
  – again bounds are set by support of *h(x)*



One more convolution

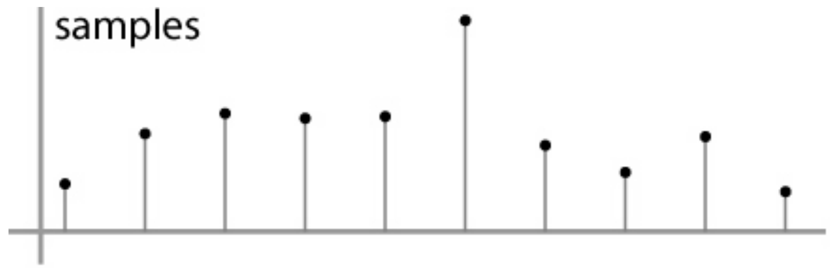
- Continuous–discrete convolution

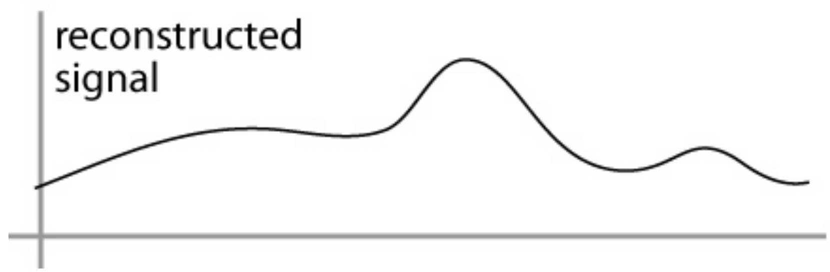
$$g(x) = \sum_i c[i] f(x - i)$$

$$g(x, y) = \sum_{i,j} c[i, j] f(x - i, y - j)$$

  – used for reconstruction and resampling

Continuous-discrete convolution



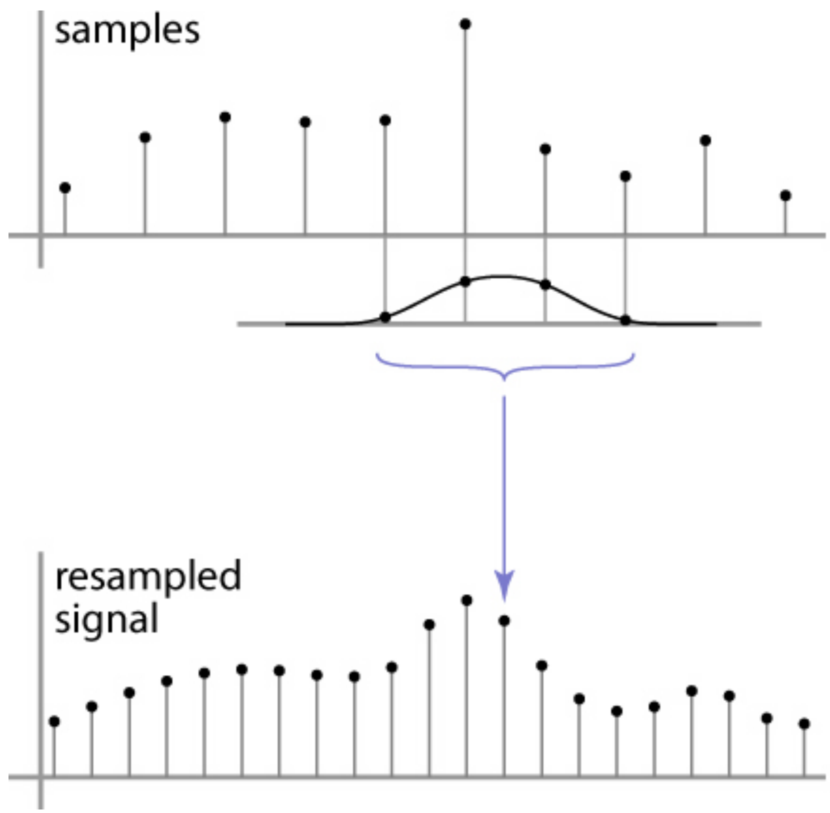


Resampling

- Changing the sample rate
  - in images, this is enlarging and reducing
- Creating more samples:
  - increasing the sample rate
  - “upsampling”
  - “enlarging”
- Ending up with fewer samples:
  - decreasing the sample rate
  - “downsampling”
  - “reducing”

Resampling

- Reconstruction creates a continuous function
  - forget its origins, go ahead and sample it

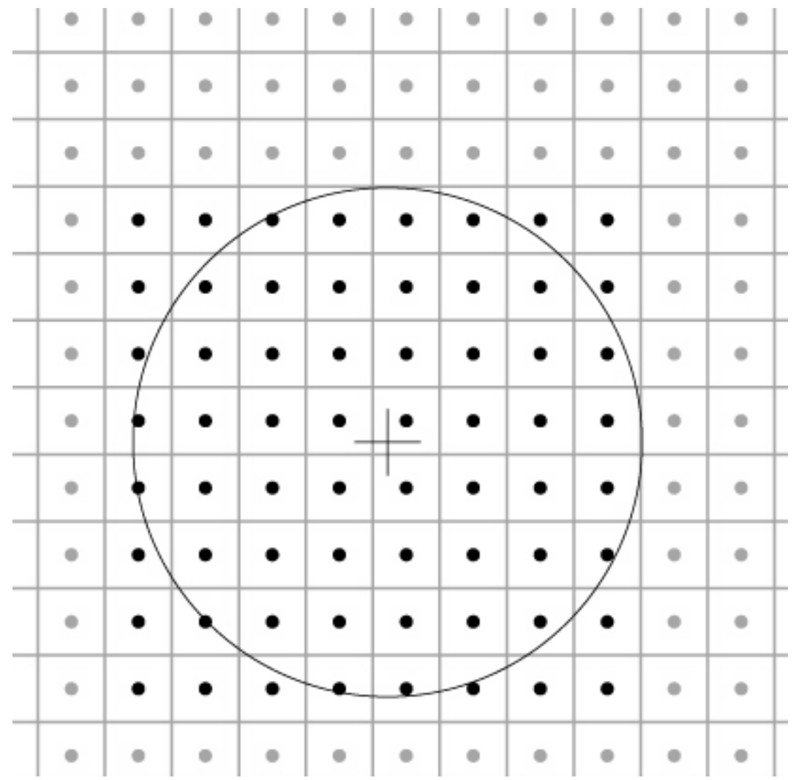


Cont.–disc. convolution in 2D

- same convolution—just two variables now

$$g(x, y) = \sum_{k,l} h(x - k, y - l) f[k, l]$$

  - loop over nearby pixels, average using filter weight
  - looks like convolution filter, but offsets are not integers and filter is continuous
  - remember placement of filter relative to grid is variable



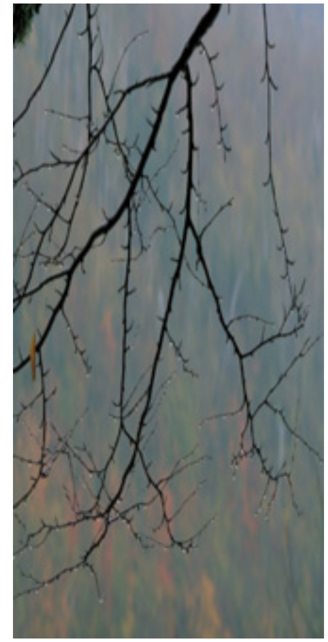
Separable filters for resampling

- just as in filtering, separable filters are useful
  - separability in this context is a statement about a continuous filter, rather than a discrete one:

$$h(x, y) = h_x(x) h_y(y)$$

- resample in two passes, one resampling each row and one resampling each column
- intermediate storage required: product of one dimension of src. and the other dimension of dest.
- same yucky details about boundary conditions

[Philip Greenspun]



two-stage resampling using a
separable filter

A gallery of filters

- Box filter
  - Simple and cheap
- Tent filter
  - Linear interpolation
- Gaussian filter
  - Very smooth antialiasing filter
- B-spline cubic
  - Very smooth
- Catmull-rom cubic
  - interpolating
- Mitchell-Netravali cubic
  - Good for image upsampling

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

Yucky details

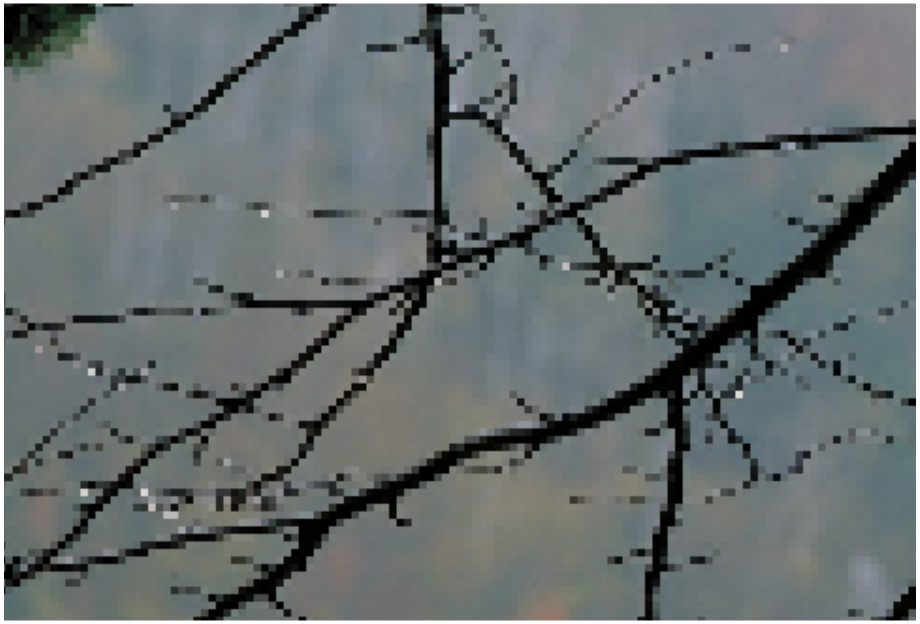
- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge
    - vary filter near edge

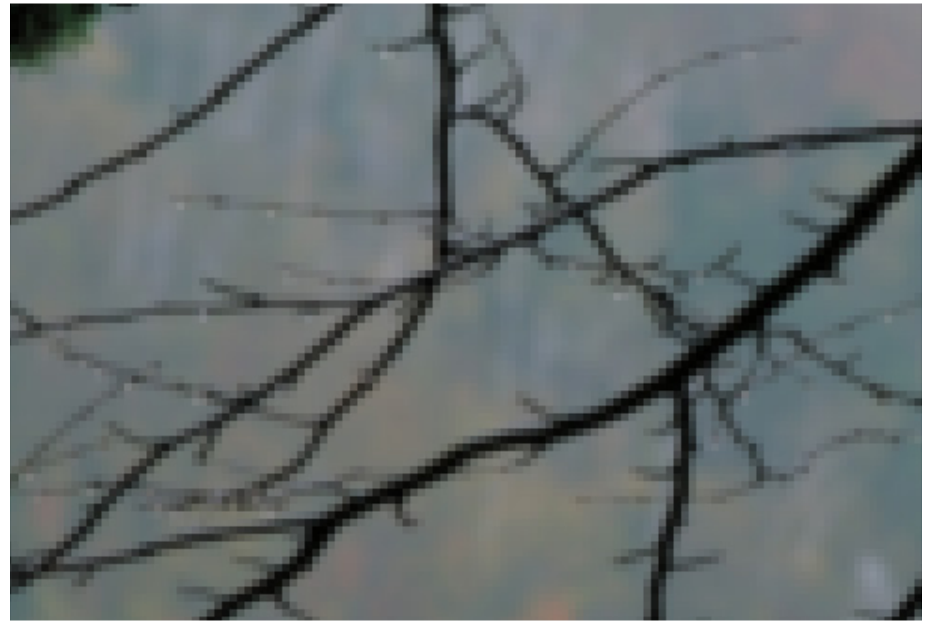
Reducing and enlarging

- very common operation
  - devices have differing resolutions
  - applications have different memory/quality tradeoffs
- also very commonly done poorly
- simple approach: drop/replicate pixels

[Philip Greenspun]

1000 pixel width





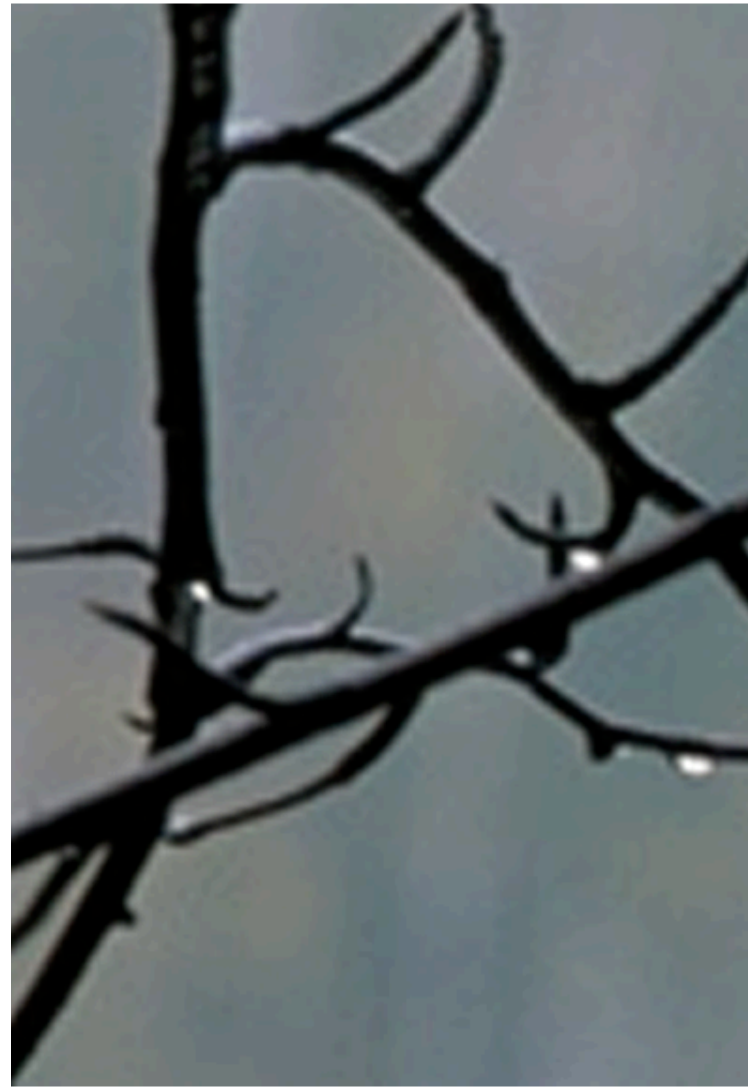
[Philip Greenspun]

by dropping pixels

gaussian filter

250 pixel width

box reconstruction filter



bicubic reconstruction filter

[Philip Greenspun]

4000 pixel width

Types of artifacts

- garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- moiré patterns

[Hearn & Baker cover]

600ppi scan of a color halftone image

by dropping pixels

gaussian filter

[Hearn & Baker cover]

downsampling a high resolution scan

Types of artifacts

- garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- moiré patterns
  - caused by repetitive patterns in input
  - produce low-frequency artifacts; highly visible
- these artifacts are called *aliasing*
  - why is beyond our scope for now
    - find out in CS467 or a signal processing class

