COMPSCI 590.1: Graph Algorithms	February 19, 2015
Lecture 13	
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1 Overview

In this lecture, we will give an introduction of background knowledge about Spectral Graph Algorithm

2 Prelimiaries

2.1 Notations

- *Matrices :* $\mathbb{M} \subset \mathbb{R}^{n\times n}$
- *rank(M) :* number of linearly independent columns of M (dimension of the matrix space)
- *null(M) :* $null(M) = \{x \in \mathbb{R}^n : Mx = 0\}$, $null(M) + rank(M) = n$
- *det(M) :* $det(M) = \sum_{\delta\in S}(-1)^{inv(\delta)}\prod_{i=1}^{n} m_{i\delta(i)}$, S: all permutation of $i = 1,2\dots,n$; inv(δ): number of crossings.

2.2 Eigenvalue and Eigenvectors

Definition 1 (Eigenvalue and Eigenvectors)**.** *Given matrix $A \in \mathbb{R}^{n\times n}$, for any non-zero vector $x \in \mathbb{R}^n$, s.t. $Ax = \lambda x$. We say λ is the eigenvalue associated with the eigenvector x.*

Based on the definition of eigenvalue and eigenvector, we have

$$\begin{aligned} & Ax = \lambda x \\ \Rightarrow\ & (A-\lambda\mathbb{I})x = 0 \\ \Rightarrow\ & rank(A-\lambda\mathbb{I}) < n \\ \Rightarrow\ & det(A-\lambda\mathbb{I}) = 0 \end{aligned}$$

Since $det(A-\lambda\mathbb{I})$ is the characteristic polynomial of degree n, the eigenvalues are roots of characteristic polynomial.

Theorem 1 (Spectral Theorem)**.** *If A is real symmetric, then there exists n real eigenvalues and a set of n orthonormal eigenvectors of A. If $rank(A) = r < n$, then its eigenvectors can be partitioned into two sets: (1) spanning the null space of A (n-r, eigenvalue = 0); (2) spanning the column space of A (r)*

$$\begin{pmatrix} 0 & & & & \\ & 0 & & 1 & \\ & & 0 & & \\ & 1 & & 0 & \\ & & & & 0 \end{pmatrix}$$

(a) Complete Graph

$$\begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$$

(b) Bipartite Graph

Figure 1: Adjacency Matrix

2.3 Diagonalization

Definition 2 (diagonalization)**.** *For all real symmetric matrix A, $A = LDL^{-1} = LDL^T$, L is orthonormal eigenvectors and D is a diagonal matrix with eigenvalues.*

Lemma 2. *The trace of matrix A $tr(A) = \sum_{i=1}^{n} \lambda_i$, λ_i are eigenvalues.*

Proof. Since $A =: LDL^{-1}$, we have

$$tr(A) = tr(LDL^{-1}) = tr(DL^{-1}L) = tr(D) = \sum_{i=1}^{n} \lambda_i$$

□

3 Graph Related Matrix

3.1 Adjacency Matrix

Definition 3 (Adjacency Matrix)**.** *Given a graph $G = (V, E)$, the adjacency matrix of G is defined as $A = [a_{ij}]_{1 \leq i,j \leq n}$, where $a_{ij} = 1$ iff $(i, j) \in E$*

For example, the adjacency matrix of a complete graph is in Figure 1(a), which can be represented as $\mathbf{1} - \mathbb{I}$. Now we compute the eigenvalues of it.

$$\begin{aligned} & (\mathbf{1} - \mathbb{I})x = \lambda x \\ \Rightarrow \quad & \mathbf{1}x = (\lambda + 1)x \end{aligned}$$

We know the eigenvalues of $\mathbf{1}$ is $n-1$ multiplicity of 0 and the last eigenvalue of n with eigenvector $\mathbf{1}$. Therefore, the eigenvalues of $\mathbf{1} - \mathbb{I}$ is $n-1$ multiplicity of -1 and the last eigenvalue equals $n-1$.

Theorem 3. *A graph is bipartite iff the eigenvalues of A occur in pairs $\lambda, -\lambda$.*

Proof. Suppose G is bipartite and λ has multiplicity k. The adjacency matrix A has the form in Figure 1(b). Then we have

$$\begin{aligned} & A(x,y)^T = \lambda(x,y)^T \\ \Rightarrow \quad & B^T y = \lambda x, Bx = \lambda y \end{aligned}$$

Also, we can get

$$A(x,-y)^T = (-B^T y, Bx)^T = (-\lambda x, \lambda y)^T = -\lambda(x,-y)^T$$

Thus, $-\lambda$ is also an eigenvalue of A.

On the other hand, suppose k is odd, we have

$$\begin{aligned} & Ax = \lambda_i x \\ \Rightarrow \quad & A^k x = \lambda_i^k x \\ \Rightarrow \quad & tr(A^k) = \sum_{i=1}^{n} \lambda_i^k = 0 \end{aligned}$$

Also, A is non-negative leads to A^k is non-negative. Therefore, $a_{ii}^k = 0$, $\forall\ i$, which means there is no length k cycle in G. G is bipartite. □

3.2 Laplacian Matrix

Definition 4 (Laplacian Matrix)**.** *Given a graph $G = (V,E)$, the Laplacian matrix of G is defined as $L = D - A$, where A is the adjacency matrix of G and D is a diagonal matrix with d_{ii} = degree of i.*

Laplacian is a positive semidefinite matrix (PSD matrix), which means $x^t Lx \geq 0$, $\forall\ x$.

Theorem 4. *For real symmetric matrices A, the following statements are equivalent:*

(a) $\forall\ x$, $x^T Ax \geq 0$

(b) $\lambda_i \geq 0$, λ_i *is any eigenvalue of A*

(c) $A = B^T B$

Proof. (a) $\Rightarrow$ (b)

$$\begin{aligned} & Ax = \lambda x \\ \Rightarrow \quad & x^T Ax = x^T \lambda x = \lambda x^T x = \lambda < x,x > \\ \Rightarrow \quad & \lambda < x,x > \geq 0 \\ \Rightarrow \quad & \lambda \geq 0 \end{aligned}$$

(b) $\Rightarrow$ (c)

$$\begin{aligned} A &= LDL^T \\ &= L\sqrt{D}\sqrt{D}L^T \\ &= LD^{1/2}D^{1/2}L^T \\ &= ((D^{1/2}L^T)^T D^{1/2}L^T = B^T B \end{aligned}$$

(c) $\Rightarrow$ (a)

$$\begin{aligned} & A = B^T B \\ \Rightarrow \quad & x^T AX = x^T B^T Bx = (Bx)^T Bx =< Bx, Bx > \geq 0 \end{aligned}$$

□

4 Summary

In this lecture, we give a brief introduction of the prelimiaries and some graph related matrix about Spectral Graph Algorithm.