Lecture 14

Lecturer: Debmalya Panigrahi *Scribe: Abhishek K Dubey*

1 Overview

In the last lecture, we discussed the electrical flows and proved a characteristic property of such flows. We showed that the electrical flows are energy minimizer among all flows. We also formulated the electrical flow problem as a linear system of equations and argued that such linear systems can be solved using laplacian system solvers in $\tilde{O}$(m) [Sim13]. In this lecture, we will see that how the electrical flows can be used to approximate the max-flow problem. The algorithm is due to Christiano et al. [CKM$^+$11] which uses the electrical flows to find the $\varepsilon-$approximate solution of max-flow problem in $\tilde{O}(m^{4/3}/\varepsilon)$. The algorithm iteratively uses a (ε, δ) flow-gadget to find a flow and the average of such flows gives us a ε-approximate solution. The complete algorithm is included in section 3. Before proceeding to the algorithm, we have formally defined the problem in the section 2. Finally, the correctness proof of the algorithm is included in section 5, followed by a brief summary in section 6.

2 Problem definition

Given a undirected graph G(V,E) and flow value F, the objective is to find the flow f from s to t such that the flow on an edge does not exceed its capacity and the incoming flow to all nodes is equal to out going flow except node s and t and the flow value satisfies the following two constraints

1. f $\geq$ (1-ε)F, if F $\leq F^*$
2. f $\geq$ (1-ε)F or output "fail", if F $> F^*$

where F^* is the maximum flow in the network.

3 Algorithm

The algorithm described in this section uses an (ε, δ) flow-gadget. The gadget has following three properties. We have included a construction of such gadget in section 4.
Given a $G = (V,E)$, $W : E \rightarrow r+$ and F: target flow value as input, an (ε, δ) flow-gadget finds a flow f such that

1. f(e) $\leq \delta$
2. $\sum_{e \in E} w(e) f(e) \leq ||w||_1 (1+\varepsilon)$
3. $|f| = F$

Assuming that such gadget can be devised in $\tilde{O}$(m) time, it is easy to see that such algorithm has a $\tilde{O}(m^{3/2}/\varepsilon)$ complexity, which can be further improved to $\tilde{O}(m^{4/3}/\varepsilon)$ using more complex analysis (not included in this lecture).

begin

$w_e^1 \leftarrow 1$

for *i = 1 to N* **do**

-find flow f^1 using the (ε, δ) flow-gadget

-update weights

$w_e^{i+1} \leftarrow w_e^i(1 + \frac{\varepsilon}{\delta} f_e^i)$

return $\bar{f} = \frac{\sum_i^N f^i}{N}$

where, N and δ is chosen as $N = \frac{2\delta ln(m)}{\varepsilon^2}$ and $\delta = \sqrt{\frac{3(1+\varepsilon)m}{\varepsilon}}$

4 (ε,δ) flow-gadget

In this section, we have provided a construction for an (ε,δ) flow-gadget. Let us consider an electrical gadget which generates the electrical flow with the following resistances $r_e = w_e + \frac{\varepsilon||w||_1}{3m}$. Now we will show that this gadget satisfies the three desired properties mentioned in above section. Let denote the OPT flow using f^* and the electrical flow using f.

Now,

$$\begin{aligned} \xi(f^*) &= \sum_{e\in E} f_e^{*2} r_e &(1)\\ &\leq \sum_{e\in E} r_e \text{f} \\ &= (1+\frac{\varepsilon}{3})||w||_1 \\ &\leq (1+\varepsilon)||w||_1 &(2) \end{aligned}$$

As the energy of the electrical flow is less than all flows,

$$\sum_{e\in E} f_e^2 (w_e + \frac{\varepsilon||w||_1}{3m}) \leq (1+\varepsilon)||w||_1 \quad (3)$$

The first term of the inequality (3) gives us,

$$\sum_{e\in E} f_e^2 w_e \leq (1+\varepsilon)||w||_1 \quad (4)$$

Now using the cauchy-schwarz inequality,

$$\begin{aligned} (\sum_{e\in E} f_e w_e)^2 &\leq (\sum_{e\in E} f_e^2 w_e)(\sum_{e\in E} w_e) \\ &\leq (1+\varepsilon)||w||_1^2 \qquad \text{using (4)} \\ \sum_{e\in E} f_e w_e &\leq \sqrt{(1+\varepsilon)}||w||_1 \\ &\leq (1+\varepsilon)||w||_1 &(5) \end{aligned}$$

This inequality shows that the electrical gadget constructed above follow the property 2.

Similarly, the second term of the inequality (3) gives us,

$$\begin{aligned}
\sum_{e \in E} \frac{f_e^2 \varepsilon ||w||_1}{3m} &\leq (1+\varepsilon)||w||_1 \\
\sum_{e \in E} f_e^2 &\leq \frac{3(1+\varepsilon)m}{\varepsilon} \\
\max f_e &\leq \sqrt{\frac{3(1+\varepsilon)m}{\varepsilon}} = \delta \qquad (6)
\end{aligned}$$

which shows that the electrical gadget also follow the property 1.

5 Correctness proof of the algorithm

In this section, we will show the correctness of the algorithm by showing that $\bar{f}$ is 1+ O(ε), i.e., the algorithm generates a ε-approximation.

As the edge weights are updated using $w_e^{i+1} \leftarrow w_e^i(1 + \frac{\varepsilon}{\delta} f_e^i)$ in each iteration,

$$\begin{aligned}
w_e^N &\leq w_e^0 \Pi_1^N \exp \frac{\varepsilon f_e^i}{\delta} \\
&= \exp \frac{\varepsilon}{\delta} \sum_1^N f_e^i \\
&= \exp \frac{\varepsilon}{\delta} N \bar{f}_e \qquad (7)
\end{aligned}$$

With the same reasoning as above,

$$\begin{aligned}
\sum_{e \in E} w_e^{i+1} &= \sum_{e \in E} w_e^i (1 + \frac{\varepsilon}{\delta} f_e^i) \\
&= ||w^i||_1 + \frac{\varepsilon}{\delta}(1+\varepsilon)||w^i||_1 \\
&= (1 + \frac{\varepsilon}{\delta}(1+\varepsilon))||w^i||_1 \qquad (8)
\end{aligned}$$

Using (8),

$$\begin{aligned}
||w^N||_1 &= (1 + \frac{\varepsilon}{\delta}(1+\varepsilon))^N m \\
&= m \exp \frac{\varepsilon}{\delta}(1+\varepsilon)N \qquad (9)
\end{aligned}$$

Also we know that

$$
\begin{aligned}
W_e^N &\leq ||W||_1^N \\
\frac{\varepsilon}{\delta}N\bar{f}_e &\leq ln(m) + \frac{\varepsilon}{\delta}(1+\varepsilon)N & \text{Using (7) and (9)} \\
\bar{f}_e &\leq (1+\varepsilon) + \frac{ln(m)}{\frac{N\varepsilon}{\delta}} \\
&= 1+\varepsilon+\frac{ln(m)}{\frac{2\delta ln(m)}{\varepsilon^2}\frac{\varepsilon}{\delta}} \\
&= 1+\frac{3\varepsilon}{2} \\
&= 1+O(\varepsilon) & (10)
\end{aligned}
$$

This completes the correctness proof.

6 Summary

In this lecture, we discussed the ε-approximation max-flow algorithm developed by Christiano et al. The algorithm uses the electrical flows to find the solution in $\tilde{O}(m^{4/3}/\varepsilon)$. The algorithm is an iterative algorithm which uses an electrical gadget to find the flows and finally take the average of flows to obtain the solution. The correctness of the algorithm is guaranteed by the properties of the gadget.

References

[CKM+11] Paul Christiano, Jonathan A Kelner, Aleksander Madry, Daniel A Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 273–282. ACM, 2011.

[Sim13] Olivia Simpson. Fast linear solvers for laplacian systems. 2013.