Game Theory

Catherine Moon
csm17@duke.edu

With thanks to Ron Parr and Vince Conitzer for some contents

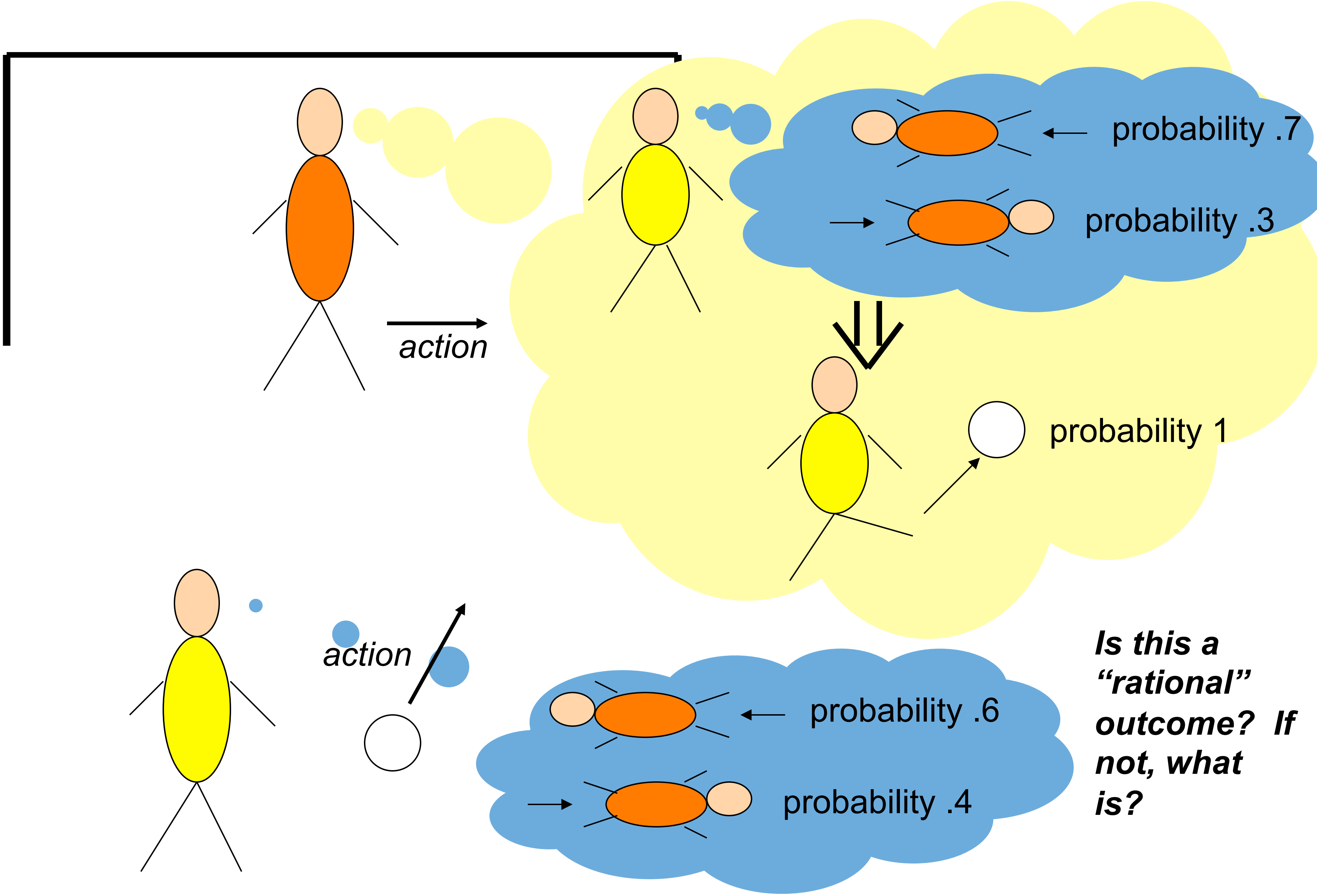
What is Game Theory?

- Settings where multiple agents each have different preferences and set of actions they can take

- Each agent's utility (potentially) depends on all agents' actions
  - What is optimal for one agent depends on what other agents do!

- Game theory studies how agents can rationally form beliefs over what other agents will do, and (hence) how agents should act

Penalty Kick Example



Is this a "rational" outcome? If not, what is?

Overview

- Zero-sum games from Adversarial Search lecture
  - Minimax, alpha-beta pruning

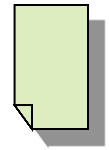
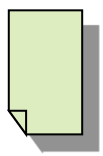
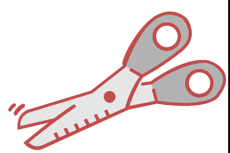
- General-sum games

- Normal form vs. Extensive form games
  - Table specifying action-payoff vs. game tree with sequence of actions (and information sets)

- Solving games: dominance, iterated dominance, mixed strategy, Nash Equilibrium

Rock-paper-scissors (zero-sum game)

Column player aka. player 2 (simultaneously) chooses a column

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Row player aka. player 1 chooses a row

A row or column is called an action or (pure) strategy

Row player's utility is always listed first, column player's second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

General-sum games

- You could still play a minimax strategy in general-sum games
  - pretend that the opponent is only trying to hurt you!
- But this is not rational:

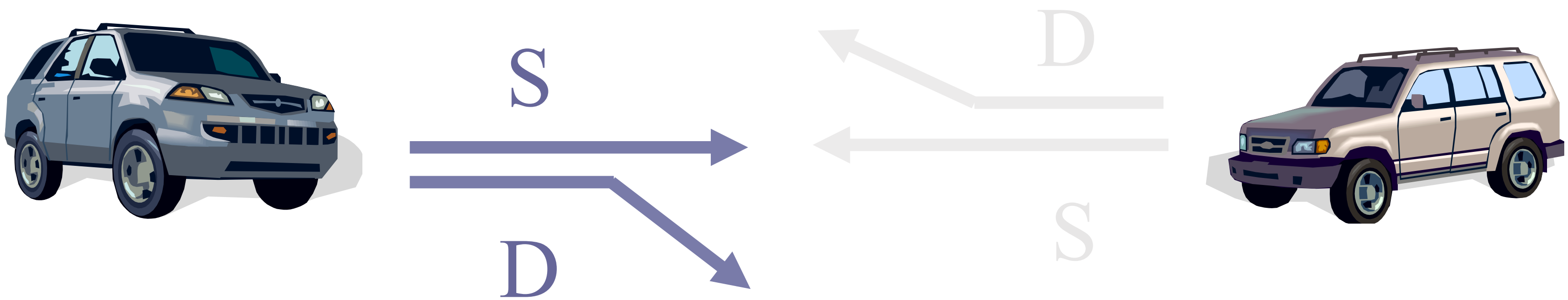
0, 0	3, 1
1, 0	2, 1

not zero-sum

- If Column was trying to hurt Row, Column would play Left, so Row should play Down
- In reality, Column will play Right (strictly dominant), so Row should play up

Chicken

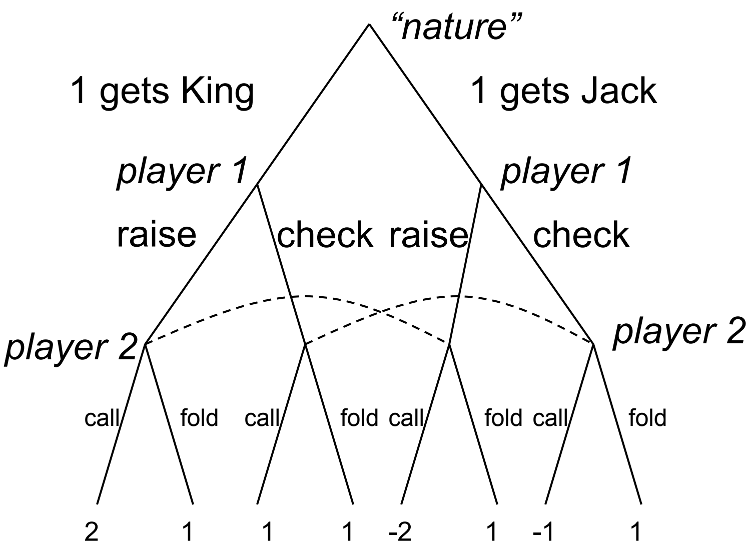
- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die



	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

not zero-sum

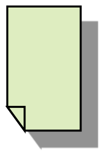
A "poker-like" game



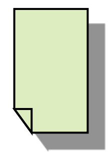
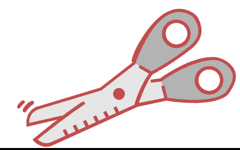
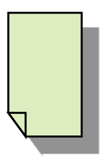
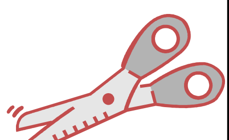
	cc	cf	fc	ff
rr	0, 0	0, 0	1, -1	1, -1
rc	.5, -.5	1.5, -1.5	0, 0	1, -1
cr	-.5, .5	-.5, .5	1, -1	1, -1
cc	0, 0	1, -1	0, 0	1, -1

Rock-paper-scissors – Seinfeld variant



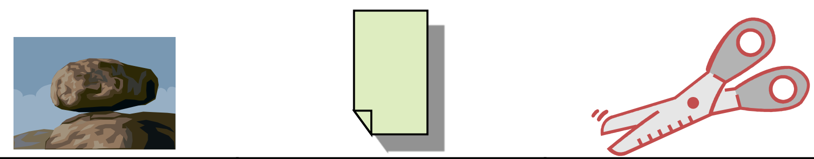
MICKEY: All right, rock beats paper!
(Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

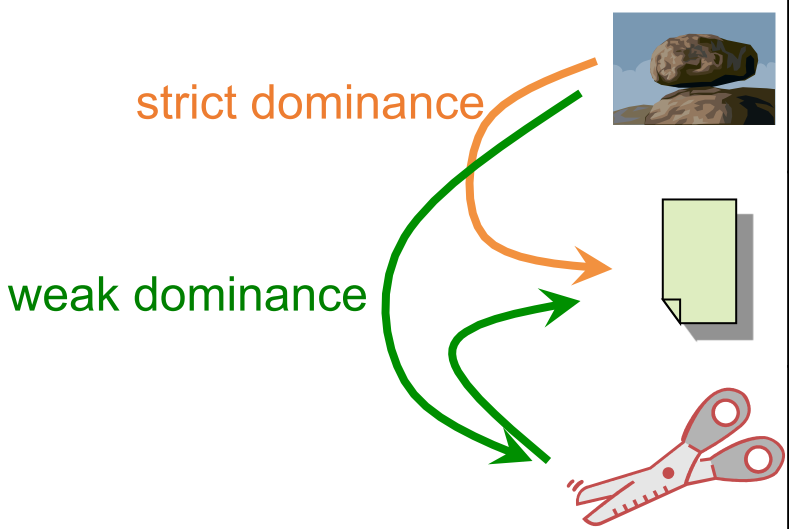
			
	0, 0	1, -1	1, -1
	-1, 1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Dominance

- Player i's strategy s_i **strictly dominates** s_i' if
  - for any s_{-i}, $u_i(s_i , s_{-i}) > u_i(s_i', s_{-i})$
- s_i **weakly dominates** s_i' if
  - for any s_{-i}, $u_i(s_i , s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some s_{-i}, $u_i(s_i , s_{-i}) > u_i(s_i', s_{-i})$

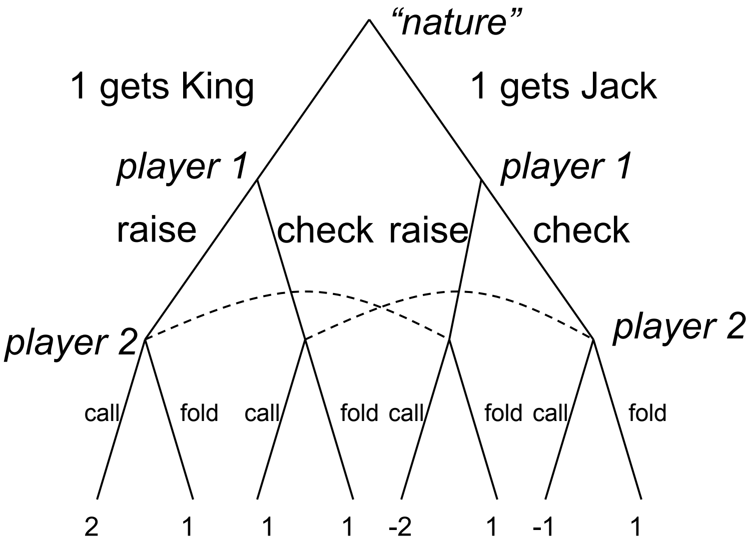
-i = "the player(s) other than i"





0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

Back to the poker like game



	cc	cf	fc	ff
rr	0, 0	0, 0	1, -1	1, -1
rc	.5, -.5	1.5, -1.5	0, 0	1, -1
cr	-.5, .5	-.5, .5	1, -1	1, -1
cc	0, 0	1, -1	0, 0	1, -1

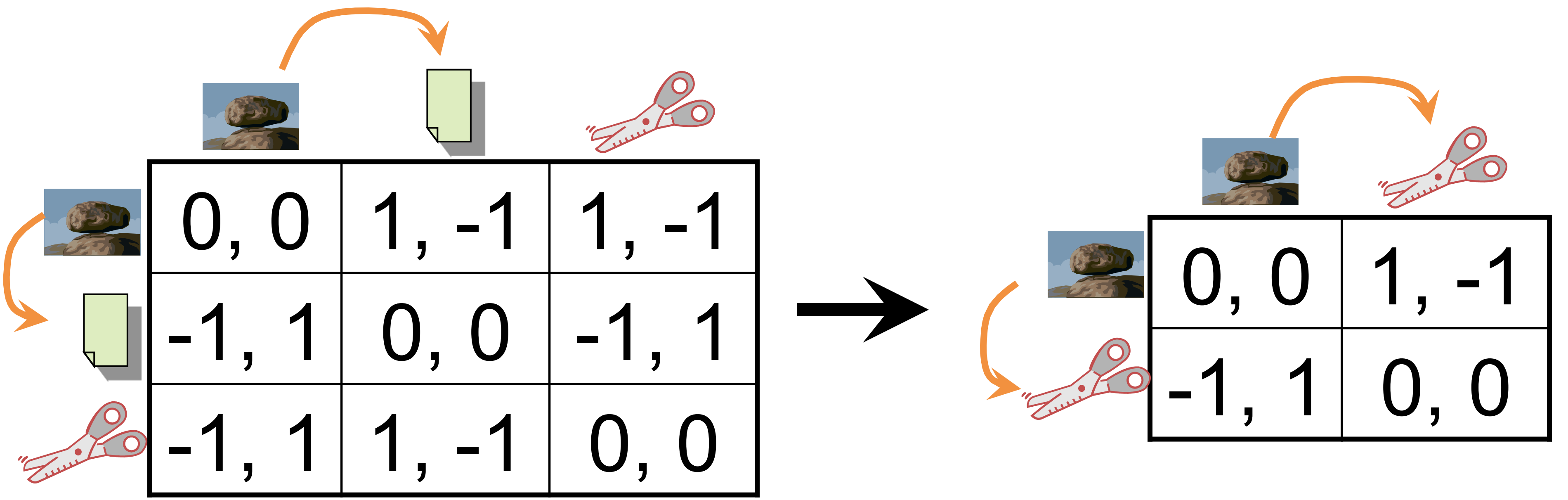
Prisoner's Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

	confess	don't confess
confess	-2, -2	0, -3
don't confess	-3, 0	-1, -1

Iterated Dominance

- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld's RPS:



0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

→

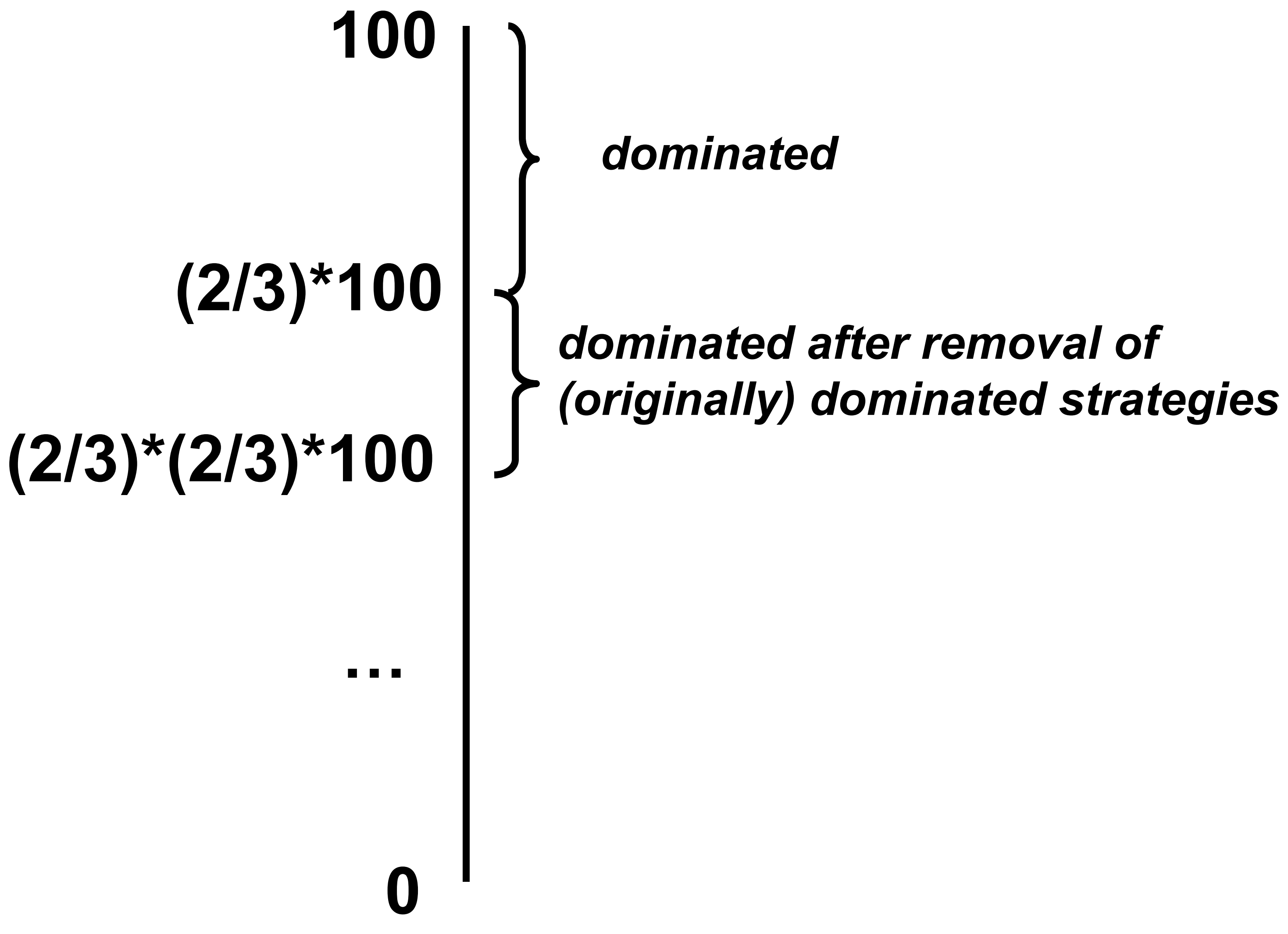
0, 0	1, -1
-1, 1	0, 0

"2/3 of the average" game

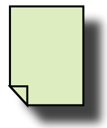
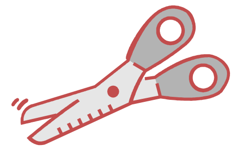
- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - 2/3 of average = 33.33
  - A is closest (|50-33.33| = 16.67), so A wins

Try?

“2/3 of the average” via dominance



Mixed strategy

- Mixed strategy for player i = probability distribution over player i’s (pure) strategies
- E.g. 1/3 , 1/3 , 1/3 
- Example of dominance by a mixed strategy:

1/2

1/2

3, 0	0, 0
0, 0	3, 0
1, 0	1, 0

Best-Response

- Let A be a matrix of player 1's payoffs
- Let s_2 be a mixed strategy for player 2
- As_2 = vector of expected payoffs for each strategy for player 1
- Highest entry indicates **best response** for player 1
- Any mixture of ties is also BR
- Generalizes to >2 players

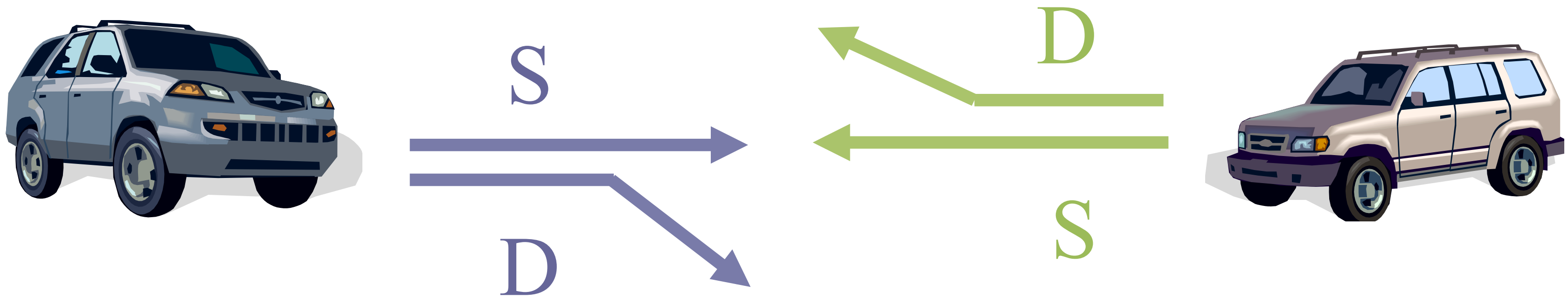
0, 0	-1, 1
1, -1	-5, -5

σ_2

Nash Equilibrium
[Nash 50]


- A vector of strategies (one for each player) = a strategy profile
- Strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Nash equilibrium if each σ_i is a best response to σ_{-i}
- Does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]

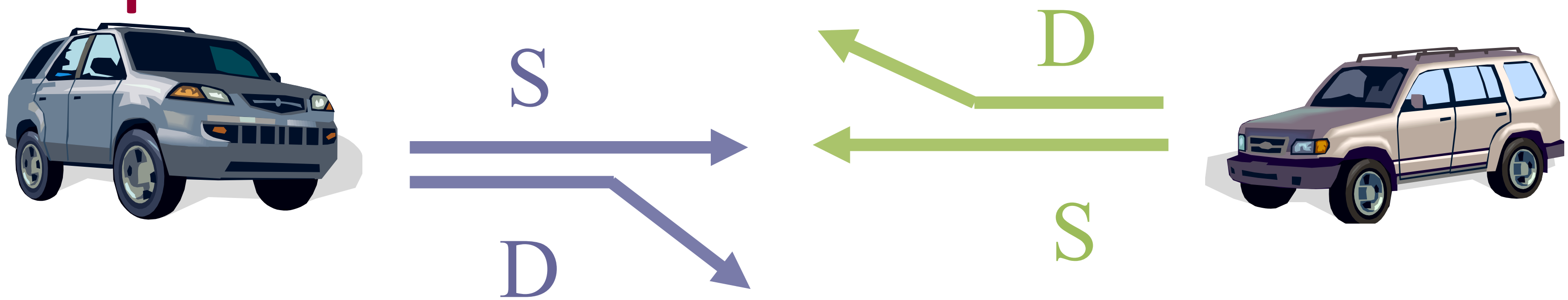
NE of "Chicken"



	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- (D, S) and (S, D) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria

Equilibrium Selection

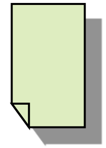
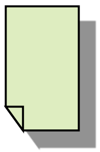


	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- (D, S) and (S, D) are Nash equilibria
- Which do you play?
- What if player 1 assumes (S, D), player 2 assumes (D, S)
- Play is (S, S) = (-5, -5)!!!

- This is the **equilibrium selection** problem

Rock-paper-scissors revisited

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

- Any pure-strategy Nash equilibria?
- But it has a mixed-strategy Nash equilibrium:

  Both players put probability 1/3 on each action
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize

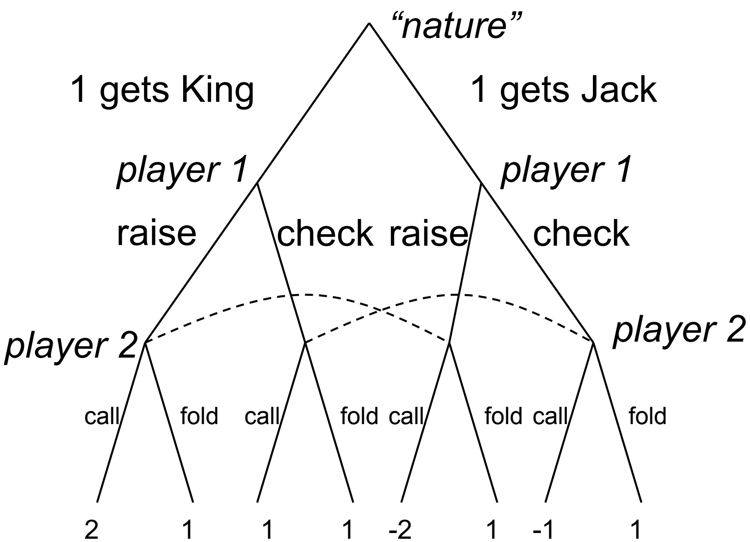
NE of "Chicken"

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- Is there a Nash equilibrium that uses mixed strategies -- say, where player 1 uses a mixed strategy?
- *If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses*
- So we need to make player 1 indifferent between D and S
- Player 1's utility for playing D = $-p^c_S$
- Player 1's utility for playing S = $p^c_D - 5p^c_S = 1 - 6p^c_S$
- So we need $-p^c_S = 1 - 6p^c_S$ which means $p^c_S = 1/5$
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player

$-p^c_S$ = probability that column player plays s

The “poker-like game” again



		2/3 cc	cf	1/3 fc	ff
1/3	rr	0, 0	0, 0	1, -1	1, -1
2/3	rc	.5, -.5	1.5, -1.5	0, 0	1, -1
	cr	-.5, .5	-.5, .5	1, -1	1, -1
	cc	0, 0	1, -1	0, 0	1, -1

- To make player 1 indifferent between rr and rc, we need:
  utility for rr = 0*P(cc)+1*(1-P(cc)) = .5*P(cc)+0*(1-P(cc)) = utility for rc
  That is, P(cc) = 2/3
- To make player 2 indifferent between cc and fc, we need:
  utility for cc = 0*P(rr)+(-.5)*(1-P(rr)) = -1*P(rr)+0*(1-P(rr)) = utility for fc
  That is, P(rr) = 1/3

Computational considerations

- Zero-sum games - solved efficiently as LP
- General sum games may require exponential time (in # of actions) to find a single equilibrium (no known efficient algorithm and good reasons to suspect that none exists)

- Some better news: Despite bad worst-case complexity, many games can be solved quickly

Extensions

- Partial information
- Uncertainty about the game parameters, e.g., payoffs (Bayesian games)
- Repeated games: Simple learning algorithms can converge to equilibria in some repeated games
- Multistep games with distributions over next states (game theory + MDPs = stochastic games)
- Multistep + partial information (Partially observable stochastic games)

- Game theory is so general, that it can encompass essentially all aspects of strategic, multiagent behavior, e.g., negotiating, threats, bluffs, coalitions, bribes, etc.