Minimum Spanning Tree 1

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1 Overview

This lecture introduces basic concepts of (MST): the cut property and the cycle property.[1] Throughout the notes, we use MST for minimal spanning tree, $w(e)$ for weight of the edge e, and $w(T)$ for weight of the tree T.

2 Minimum Spanning Tree Preliminaries

Definition 1. *Given an undirected weighted connected graph $G = (V,E)$, a spanning tree is a subgraph $G' = (V,E')$ of G, where $E' \subseteq E$, such that G' is connected and acyclic.*

Definition 2. *A minimum spanning tree (MST) is a spanning tree with minimum total weight.*

2.1 Generic Properties of Minimum Spanning Tree

2.1.1 Cut Property

Definition 3. *A cut of a graph $G = (V,E)$ is a pair of disjoint and exhaustive subsets of V. A cut determines a cut-set, which is the set of edges that have one endpoint in each subset of the pair.*

Lemma 1 (Cut Property)**.** *Given an undirected weighted connected graph $G = (V,E)$, for any $S \subseteq V$, the (strictly) lightest edge cross the cut $(S,V \setminus S)$ is included in any minimum spanning tree.*

Proof. Let T be a minimum spanning tree. Let the lightest edge cross the cut $(S,V \setminus S)$ be (u,v), where $u \in S$ and $v \in V \setminus S$. If T does not contain (u,v), we can find an edge $e \neq (u,v)$ in T which fulfills: (1) e is in the path from u to v and (2) e is an edge cross the cut $(S,V \setminus S)$. Such an edge has to exist because T is a spanning tree. We construct another spanning tree T' by deleting edge e from T and adding edge (u,v), and then T' has a smaller total weight which implies that T is not a minimum spanning tree. □

2.1.2 Cycle Property

Lemma 2 (Cycle Property)**.** *Let C be any cycle, and let f be the (unique) max weight edge belonging to C. Then any MST does not contain f.*

Proof. Let T^* be an MST. Suppose f belongs to T^*. Deleting f from T^* disconnects T^* and generates a cut $(S,V \setminus S)$. There is some other edge in the cycle C, say e, has exactly one endpoint in S. Therefore $T = T^* \setminus \{e\} \cup \{f\}$ is also a spanning tree. Since $w(e) < w(f)$, we know $w(T) < w(T^*)$, which is a contradiction. □

[1]Some of the material is from a previous note by Yilun Zhou for this course in Fall 2014.