CS 330 Discussion - Duality

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1 Dual of a Simple LP

Recall that given a linear program of the following form:

$$\min \sum_i c_i x_i$$

$$s.t. \forall j : \sum_i a_{ij} x_i \geq b_j$$

$$\forall i : x_i \geq 0$$

The dual linear program is:

$$\min \sum_j b_j y_j$$

$$s.t. \forall_i \sum_j a_{ij} y_j \leq c_i$$

$$\forall j : y_j \geq 0$$

Or, using vectors, the dual of

$$\min c^T x$$

$$s.t. Ax \geq b$$

$$x \geq 0$$

is

$$\max b^T y$$

$$s.t. A^T y \leq c$$

$$y \geq 0$$

For example, suppose we wish to take the dual of

$$\min 3x_1 + 2x_2 + 4x_3$$

$$s.t. x_1 + 2x_2 + x_3 \geq 4$$

$$3x_1 + x_2 + 2x_3 \geq 3$$

$$x_1, x_2, x_3 \geq 0$$

Then, the dual is:

$$\max 4y_1 + 3y_2$$

(one variable for each non-trivial constraint, the coefficient in the objective is the right hand side of the corresponding constraint in the primal)

$$s.t. y_1 + 3y_2 \leq 3$$

(the constraint corresponding to x_1 - for each y_j the coefficient is the coefficient of x_1 in the constraint corresponding to that t_j. The right hand side of the constraint is the multiplier of x_1 in the primal objective)

$$2y_1 + y_2 \leq 2$$

$$y_1 + 2y_3 \leq 4$$

(Obtained using the same logic for x_2, x_3)

2 Dual of Algorithm LPs

We can also take the dual of certain problems in algorithms by writing them as linear programs.

Consider the vertex cover problem. Given an graph, we want to find the smallest set of vertices such that every edge has at least one endpoint in that set. For the linear program, we'll define x_v for each vertex as an indicator variable, which is 1 if v is in our solution and 0 otherwise.

Then, finding the smallest vertex cover corresponds to:

$$\min \sum_v x_v$$

And to make sure every edge has one of its endpoints chosen:

$$s.t. \forall e = (u, v) \in E : x_u + x_v \geq 1$$

We also include the trivial constraint:

$$x_v \geq 0$$

To take the dual - we have a constraint for each edge, so in the dual we have a variable for each edge, y_e. Then, for each constraint the constant side of the inequality is 1, so all coefficients in the objective are 1, so we get:

$$\max \sum_e y_e$$

Then, we have a variable for each vertex, so in the dual we have a constraint for each vertex. Then, the variable x_v appears in the constraint for each edge adjacent to it in the primal. So, the constraint corresponding to that vertex should sum over all the variables corresponding to those edges in the dual. The coefficient of x_v in the primal objective is 1, so the inequality will have a constant 1:

$$s.t. \forall v : \sum_{e \in \delta(v)} y_e \leq 1$$

Lastly, we include the trivial constraint:

$$\forall e : y_e \geq 0$$

Note that integrally, this is the maximum matching problem. The dual can be interpreted as - find the largest set of edges, such that we do not choose more than one edge adjacent to any vertex.

3 Dual of More Complicated LPs

All linear programs can be written in the standard form shown at the top of the first page. However, we may occasionally with to take the dual of a linear program which has equalities, greater than and less than inequalities, free variables, or negative variables.

The correspondence of constraints and variables in the primal and dual still holds in this case, but there are more cases to cover. The following shows possible forms for constraints and variables in a minimization problem and what the corresponding variables and constraints in a dual maximization problem look like.

Minimization	Maximization
Objective: min $\sum_i c_i x_i$	Objective: max $\sum_j b_j y_j$
Constraint j: $\sum_i a_{ij} x_i \geq b_i$	Variable j: $y_j \geq 0$
Constraint j: $\sum_i a_{ij} x_i = b_i$	Variable j: free y_j
Constraint j: $\sum_i a_{ij} x_i \leq b_i$	Variable j: $y_j \leq 0$
Variable i: $x_i \geq 0$	Constraint i: $\sum_j a_{ij} y_j \leq 0$
Variable i: free x_i	Constraint i: $\sum_j a_{ij} y_j = 0$
Variable i: $x_i \leq 0$	Constraint i: $\sum_j a_{ij} y_j \geq 0$