This problem sheet will be updated through the course of the semester. You will have to do any six problems. The answers must be typed in LaTeX, and emailed to the instructor in a single pdf file. **No collaboration is allowed.**

Problem 1
In class, we proved a running time bound of $O(m^{3/2}\log(n^2/m)\log(nU))$ for the Goldberg-Rao maximum flow algorithm. Show that this bound (for the same algorithm) can be improved to $O(m\min(m^{1/2},n^{2/3})\log(n^2/m)\log U)$. *(Hint: Improve the running time of Dinitz's algorithm for maximum flows using blocking flows for unit graphs from $O(m^{3/2})$ to $O(m\min(m^{1/2},n^{2/3}))$. Also, obtain a tighter bound for the number of capacity scaling iterations.)*

Problem 2
Show that edge strength, conductance, and the Nagamochi-Ibaraki index are incomparable parameters, i.e., for any two parameters α_e, β_e, show that there is an example where $\alpha_e < \beta_e$.

Problem 3
In class, we proved that the number of α-minimum cuts in undirected graphs is $n^{O(\alpha)}$. Improve this bound to $\binom{n}{2\alpha}$, and show that this is tight.

Problem 4
For any undirected graph G, suppose p_e and q_e are probabilities defined on the edges such that $p_e \geq q_e$ for all edges e. Now, consider two randomly generated graphs, G_p and G_q, formed by sampling every edge e of G independently with probablity p_e and q_e, and taking $1/p_e$ and $1/q_e$ copies of each sampled edge respectively. Show that for any cut $(S,\overline{S})$ and any $\epsilon > 0$,

$$\mathbb{P}\left[G_p(S,\overline{S}) \notin (1\pm\epsilon)G(S,\overline{S})\right] \leq \mathbb{P}\left[G_q(S,\overline{S}) \notin (1\pm\epsilon)G(S,\overline{S})\right].$$

Problem 5
In class, we proved that the sum of reciprocals of connectivities of all edges is $O(n)$, while that of reciprocals of NI indices is $O(n\log n)$. Show that this gap is tight, i.e., there are graphs where the first sum is $O(n)$ while the second sum is $\Omega(n\log n)$ for *any* greedy spanning forest packing of the graph.

Problem 6
Show that any greedy spanning forest packing of a simple, unit capacity graph has at most $n-1$ forests. Next, show that this bound is tight, i.e., there exists a greedy spanning forest packing of some unit capacity, simple graph that contains exactly $n-1$ forests.

Problem 7
Show that the number of connected components of an undirected graph is equal to the multiplicity of 0 as an eigenvalue.

Problem 8
Show that the running time of the Karger-Klein-Tarjan minimum spanning tree algorithm is linear in the number of edges with high probability (as against in expectation, which you saw in lecture).

Problem 9
Let F be the set of tight edges in the primal dual algorithm for edge-weighted Steiner forest. Define

$$F' = \{e \in F : F \setminus \{e\} \text{ is not a feasible solution}\}.$$

Then, show F' is a feasible solution.

Problem 10
(a) Show that the edge-weighted Steiner tree problem is NP-hard by a reduction from the vertex cover problem.
(b) Show that the node-weighted Steiner tree problem cannot be approximated to a factor of $o(\log k)$, where k is the number of terminals, by an approximation-preserving reduction from the set cover problem.