CPS 223

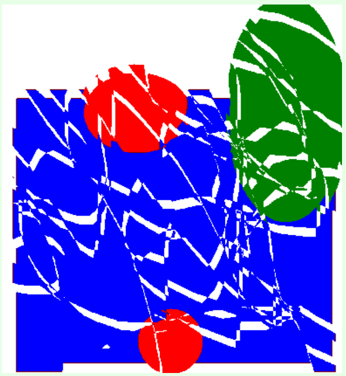
Linear Programming Duality, Reductions, and Bipartite Matching

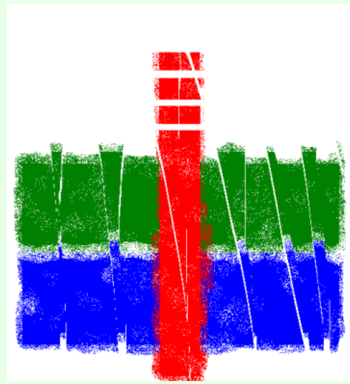
Yu Cheng

Linear Programming Duality

Example linear program

- We make reproductions of two paintings





maximize 3x + 2y

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

Solving the linear program graphically

maximize $3x + 2y$

subject to

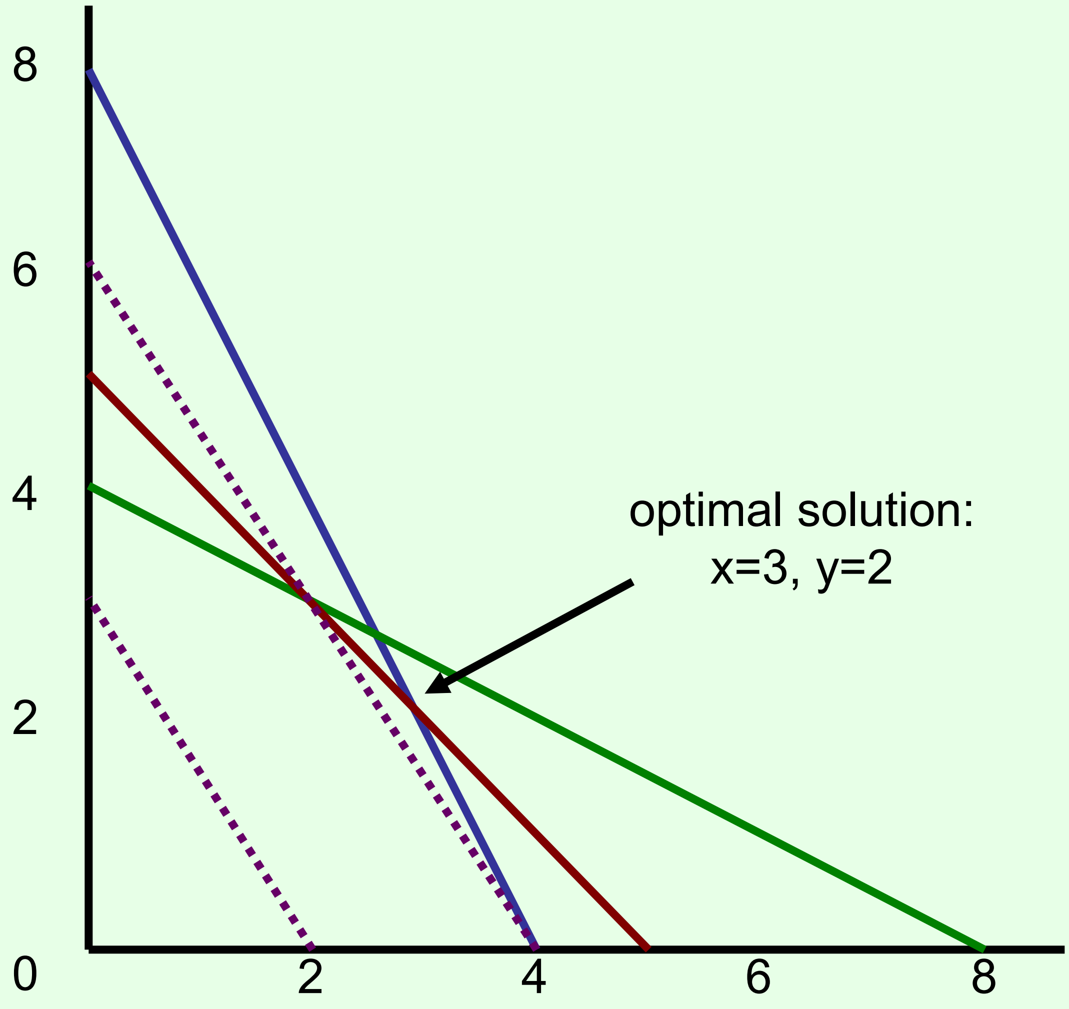
$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$



Proving optimality

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Recall: optimal solution: $x=3$, $y=2$

Solution value = 9+4 = 13

How do we prove this is optimal (without the picture)?

Proving optimality…

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

We can rewrite the blue constraint as

$2x + y \leq 8$

If we add the red constraint

$x + y \leq 5$

we get

$3x + 2y \leq 13$

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)

Linear combinations of constraints

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

$b(4x + 2y \leq 16)$ +

$g(x + 2y \leq 8)$ +

$r(x + y \leq 5)$

=

$(4b + g + r)x +$

$(2b + 2g + r)y \leq$

$16b + 8g + 5r$

$4b + g + r$ must be at least 3

$2b + 2g + r$ must be at least 2

Given this, minimize $16b + 8g + 5r$

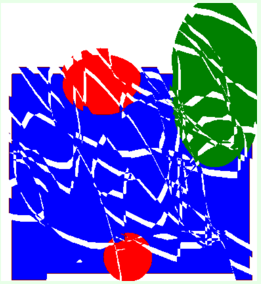
Using LP for getting the best upper bound on an LP

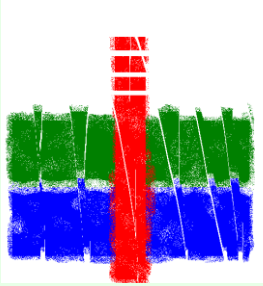
maximize $3x + 2y$	*minimize* $16b + 8g + 5r$
subject to	*subject to*
$4x + 2y \leq 16$	$4b + g + r \geq 3$
$x + 2y \leq 8$	$2b + 2g + r \geq 2$
$x + y \leq 5$	$b \geq 0$
$x \geq 0$	$g \geq 0$
$y \geq 0$	$r \geq 0$

the dual of the original program

- Duality theorem: any linear program has the same optimal value as its dual!

Another View



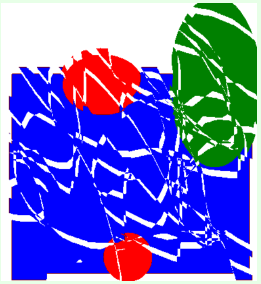


- Painting 1: 4 blue, 1 green, 1 red, sells for $30
- Painting 2: 2 blue, 2 green, 1 red, sells for $20
- We have 16 units blue, 8 green, 5 red

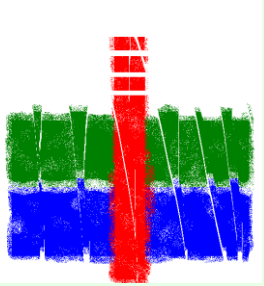
- Suppose Vince wants to buy paints from us.
- Pay $b for a unit of blue, $g for green, $r for red.
- We can choose to sell the paints, or produce paintings and sell the paintings, or both.

$$b \geq 0$$
$$g \geq 0$$
$$r \geq 0$$

$$4b + g + r \geq 3$$
$$2b + 2g + r \geq 2$$



Another View



- Vince pays \$($16b + 8g + 5r$) in total.

- We have 16 units blue, 8 green, 5 red

  - Suppose Vince wants to buy paints from us.
  - Pay \$b for a unit of blue, \$g for green, \$r for red.
  - We can choose to sell the paints, or produce paintings and sell the paintings, or both.

$$b \geq 0$$
$$g \geq 0$$
$$r \geq 0$$

$$4b + g + r \geq 3$$
$$2b + 2g + r \geq 2$$

Using LP for getting the best upper bound on an LP

primal	dual
maximize $3x + 2y$	*minimize* $16b + 8g + 5r$
subject to	*subject to*
$4x + 2y \leq 16$	$4b + g + r \geq 3$
$x + 2y \leq 8$	$2b + 2g + r \geq 2$
$x + y \leq 5$	$b \geq 0$
$x \geq 0$	$g \geq 0$
$y \geq 0$	$r \geq 0$

Duality

- Weak duality:

  Optimal value of primal ≥ Optimal value of dual

  (when primal LP is max and dual LP is min)

- We can make \$13 if we produce paintings

  Vince should pay at least as much

- Strong Duality

  Optimal value of primal = Optimal value of dual

  Vince is a good negotiator

Using LP for getting the best upper bound on an LP

primal	dual
maximize $3x + 2y$	*minimize* $16b + 8g + 5r$
subject to	*subject to*
$4x + 2y \leq 16$	$4b + g + r \geq 3$
$x + 2y \leq 8$	$2b + 2g + r \geq 2$
$x + y \leq 5$	$b \geq 0$
$x \geq 0$	$g \geq 0$
$y \geq 0$	$r \geq 0$

Reductions

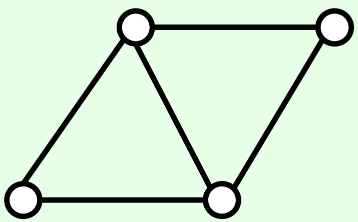
NP ("nondeterministic polynomial time")

- Recall: decision problems require a yes or no answer
- NP: the class of all decision problems such that if the answer is yes, there is a simple proof of that
- E.g., "does there exist a set cover of size k?"
- If yes, then just show which subsets to choose!

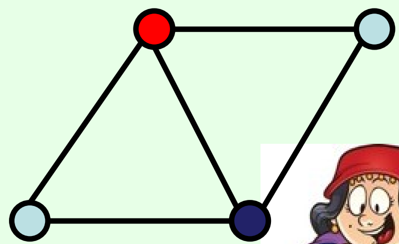
- Technically:
  - The proof must have polynomial length
  - The correctness of the proof must be verifiable in polynomial time

“Easy to verify” problems: NP

- All decision problems such that we can verify the correctness of a solution in polynomial time.



input



Prover

Verifier: OK, that is indeed a solution.

NP-hardness

- A problem is NP-hard if it is at least as hard as all problems in NP
- So, trying to find a polynomial-time algorithm for it is like trying to prove P=NP
- Set cover is NP-hard

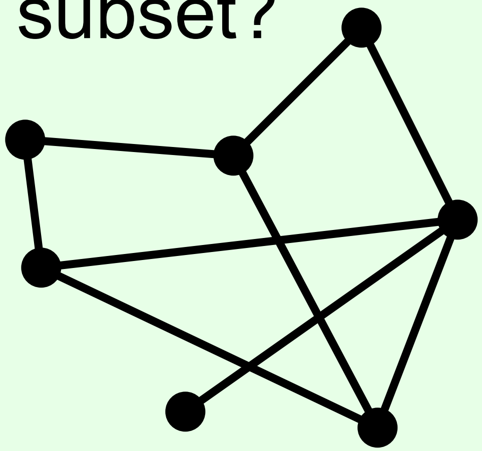
- Typical way to prove problem Q is NP-hard:
  - Take a known NP-hard problem Q'
  - Reduce it to your problem Q
    - (in polynomial time)
- E.g., (M)IP is NP-hard, because we have already reduced set cover to it
  - (M)IP is more general than set cover, so it can't be easier

Reductions

- Sometimes you can reformulate problem A in terms of problem B (i.e., reduce A to B)
  - E.g., we have seen how to formulate several problems as linear programs or integer programs
- In this case problem A is at most as hard as problem B
  - Since LP is in P, all problems that we can formulate using LP are in P
  - Caveat: only true if the linear program itself can be created in polynomial time!

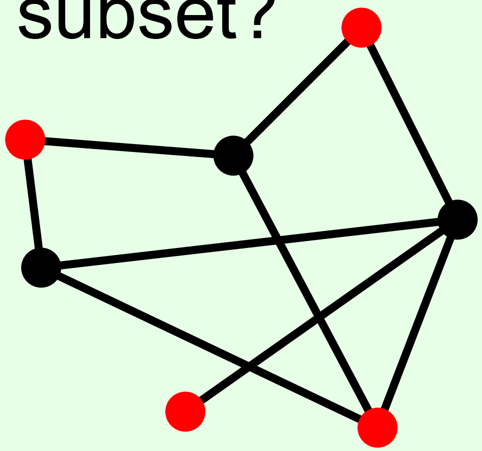
Independent Set

- In the below graph, does there exist a subset of vertices, of size 4, such that there is no edge between members of the subset?



Independent Set

- In the below graph, does there exist a subset of vertices, of size 4, such that there is no edge between members of the subset?



- General problem (decision variant): given a graph and a number k, are there k vertices with no edges between them?
- NP-complete

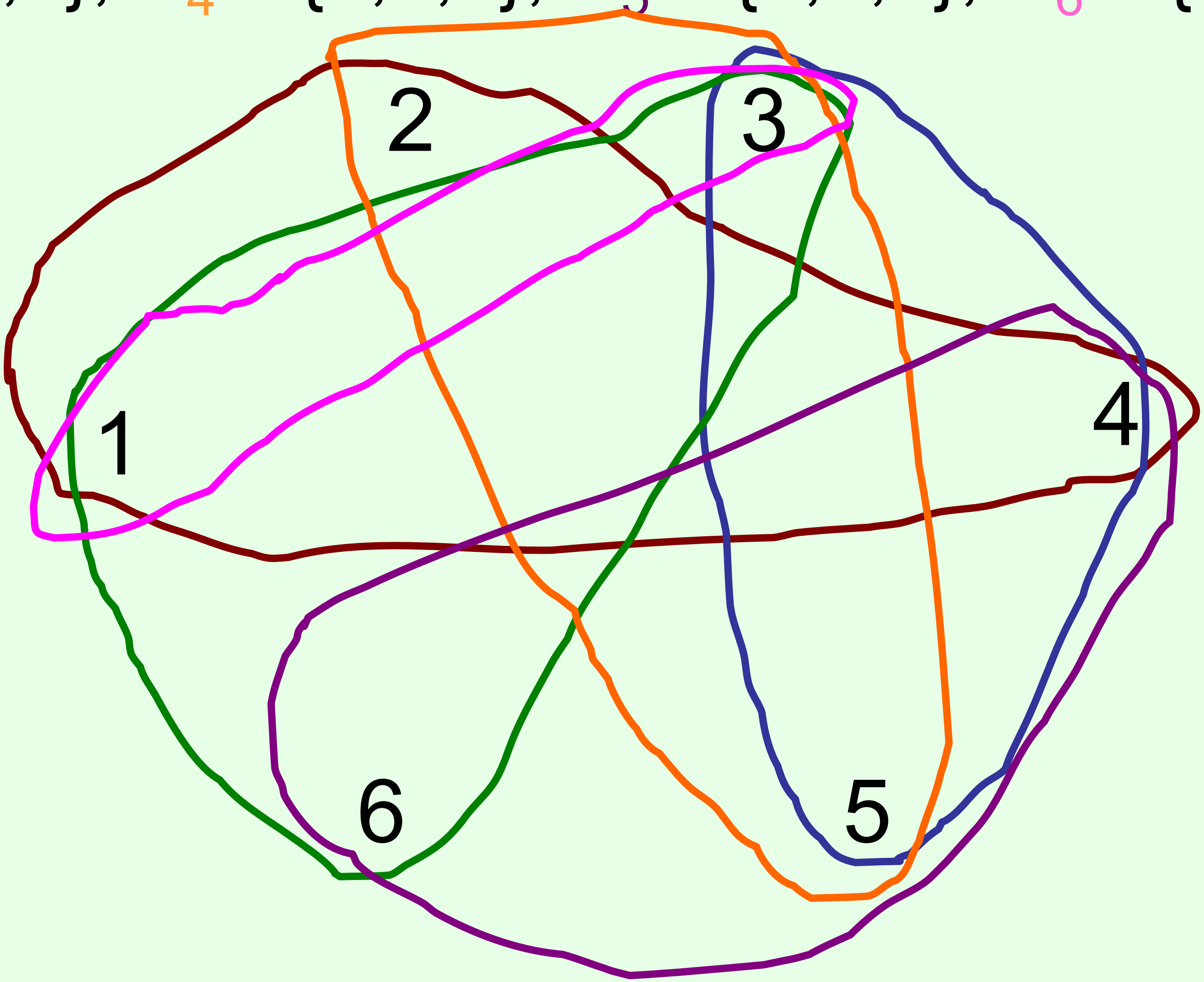
Set Cover (a computational problem)

- We are given:
  - A finite set S = {1, …, n}
  - A collection of subsets of S: S_1, S_2, …, S_m
- We are asked:
  - Find a subset T of {1, …, m} such that $U_{j \text{ in } T} S_j = S$
  - Minimize |T|
- Decision variant of the problem:
  - we are additionally given a target size k, and
  - asked whether a T of size at most k will suffice
- One instance of the set cover problem:

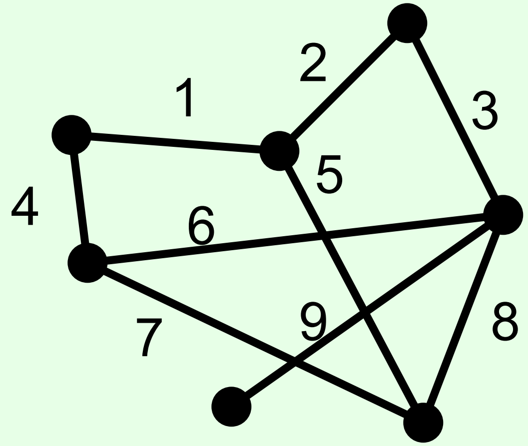
S = {1, …, 6}, S_1 = {1,2,4}, S_2 = {3,4,5}, S_3 = {1,3,6}, S_4 = {2,3,5}, S_5 = {4,5,6}, S_6 = {1,3}

Visualizing Set Cover

- $S = \{1, \ldots, 6\}$, $S_1 = \{1,2,4\}$, $S_2 = \{3,4,5\}$, $S_3 = \{1,3,6\}$, $S_4 = \{2,3,5\}$, $S_5 = \{4,5,6\}$, $S_6 = \{1,3\}$



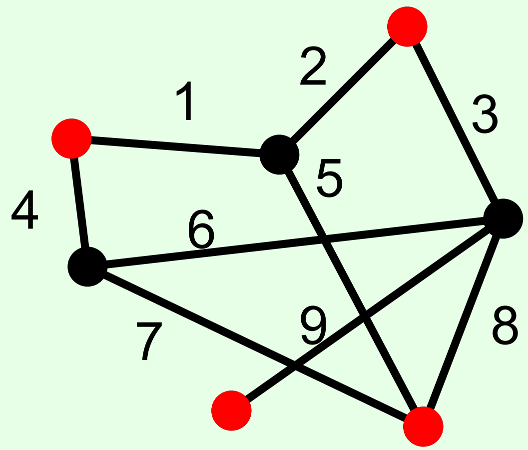
Reducing independent set to set cover



, k=4

- In set cover instance (decision variant),
  - let S = {1,2,3,4,5,6,7,8,9} (set of edges),
  - for each vertex let there be a subset with the vertex's adjacent edges: {1,4}, {1,2,5}, {2,3}, {4,6,7}, {3,6,8,9}, {9}, {5,7,8}
  - target size = #vertices - k = 7 - 4 = 3
- Claim: answer to both instances is the same (why??)

Reducing independent set to set cover



, k=4

- In set cover instance (decision variant),
  - let S = {1,2,3,4,5,6,7,8,9} (set of edges),
  - for each vertex let there be a subset with the vertex's adjacent edges: {1,4}, {1,2,5}, {2,3}, {4,6,7}, {3,6,8,9}, {9}, {5,7,8}
  - target size = #vertices - k = 7 - 4 = 3
- Claim: answer to both instances is the same (why??)
- So which of the two problems is harder?

Reductions:

To show problem Q is easy:

Q —*reduce*→ Problem known to be easy (e.g., LP)

To show problem Q is (NP-)hard:

Problem known to be (NP-)hard (e.g., set cover, (M)IP) —*reduce*→ Q

Polynomial time reductions

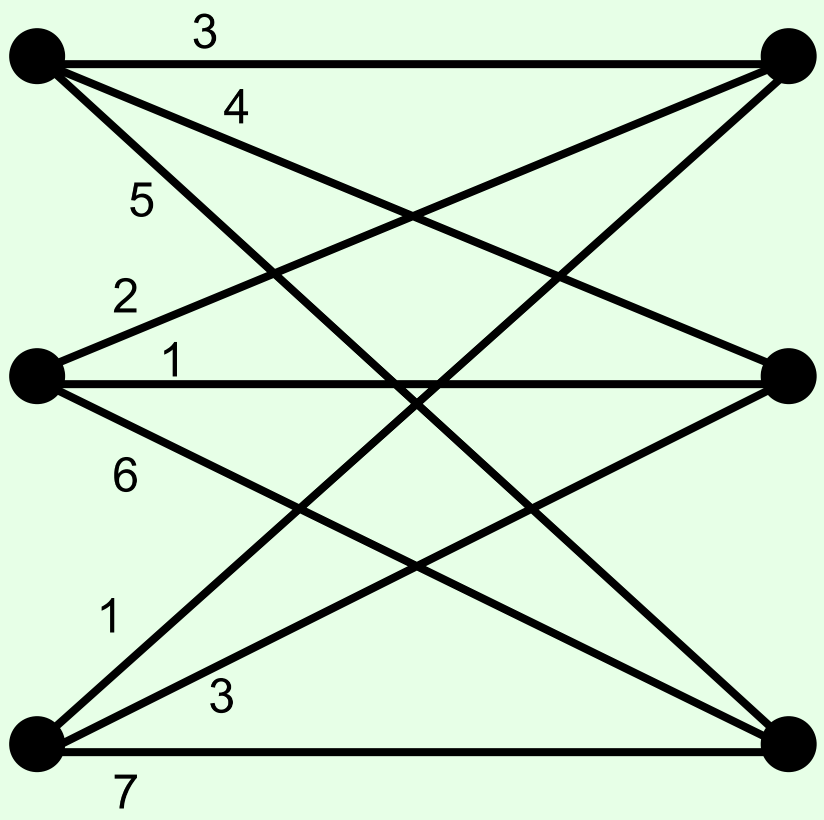
- Reduce A to B: a polynomial time algorithm that maps instances of A to instances of problem B, such that the answers are the same.

- $A \leq_p B$: B is at least as hard as A.

  If you can solve B (in poly time) then you can solve A.

Weighted Bipartite Matching

Weighted bipartite matching



- Match each node on the left with one node on the right (can only use each node once)
- Minimize total cost (weights on the chosen edges)

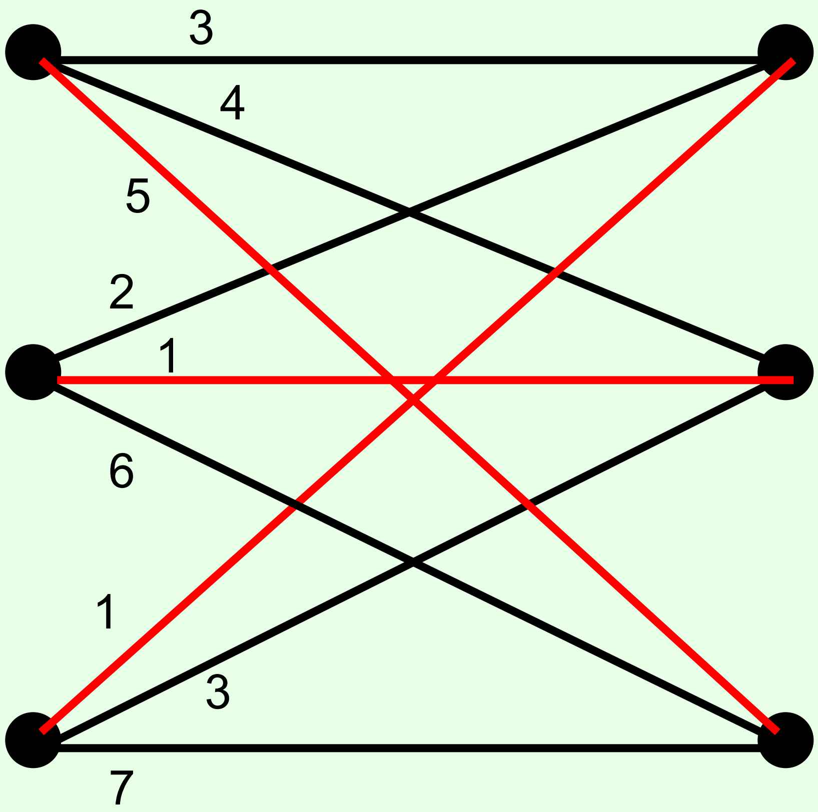
Weighted bipartite matching…

- minimize $c_{ij}\ x_{ij}$
- subject to
- for every i, $\Sigma_j\ x_{ij} = 1$
- for every j, $\Sigma_i\ x_{ij} = 1$
- for every i, j, $x_{ij} \geq 0$

- Theorem [Birkhoff-von Neumann]: this linear program always has an optimal solution consisting of just integers
  – and typical LP solving algorithms will return such a solution

- So weighted bipartite matching is in P

Weighted bipartite matching



- Match each node on the left with one node on the right (can only use each node once)
- Minimize total cost (weights on the chosen edges)