

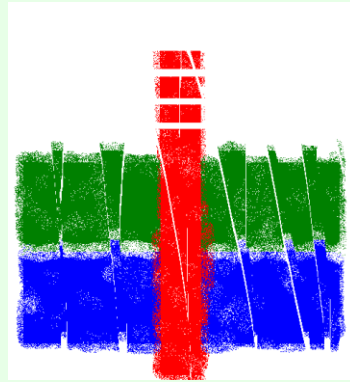
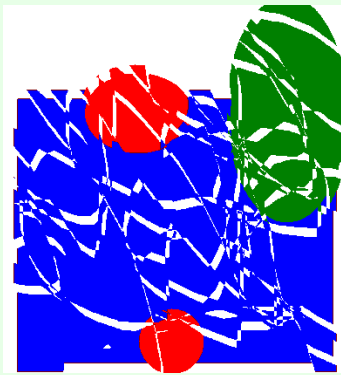
# CPS 223

Linear programming,  
integer linear programming,  
mixed integer linear programming

[conitzer@cs.duke.edu](mailto:conitzer@cs.duke.edu)

# Example linear program

- We make reproductions of two paintings



*maximize*  $3x + 2y$

*subject to*

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

- Painting 1 sells for \$30, painting 2 sells for \$20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

# Solving the linear program graphically

*maximize*  $3x + 2y$

*subject to*

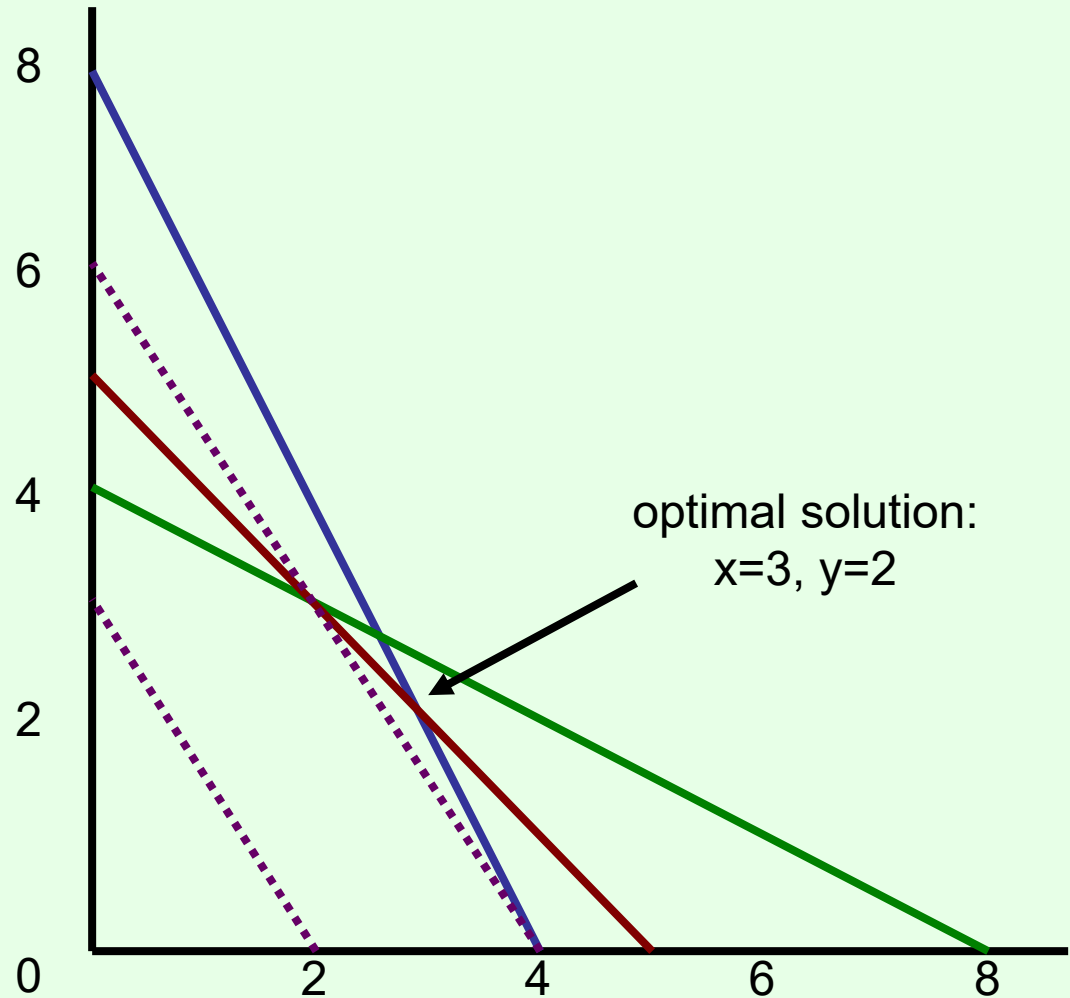
$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$



# Proving optimality

*maximize*  $3x + 2y$

*subject to*

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Recall: optimal solution:

$$x=3, y=2$$

$$\text{Solution value} = 9+4 = 13$$

How do we **prove** this is optimal (without the picture)?

# Proving optimality...

*maximize*  $3x + 2y$

*subject to*

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

We can rewrite the blue constraint as

$$2x + y \leq 8$$

If we add the red constraint

$$x + y \leq 5$$

we get

$$3x + 2y \leq 13$$

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)

# Linear combinations of constraints

maximize  $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

$$b(4x + 2y \leq 16) +$$

$$g(x + 2y \leq 8) +$$

$$r(x + y \leq 5) =$$

$$(4b + g + r)x +$$

$$(2b + 2g + r)y \leq$$

$$16b + 8g + 5r$$

$4b + g + r$  must

be at least 3

$2b + 2g + r$  must

be at least 2

Given this,

minimize  $16b +$

$$8g + 5r$$

# Using LP for getting the best upper bound on an LP

*maximize*  $3x + 2y$

*subject to*

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

*minimize*  $16b + 8g + 5r$

*subject to*

$$4b + g + r \geq 3$$

$$2b + 2g + r \geq 2$$

$$b \geq 0$$

$$g \geq 0$$

$$r \geq 0$$

the **dual** of the original program

- Duality theorem: any linear program has the same optimal value as its dual!

# Modified LP

*maximize*  $3x + 2y$

*subject to*

$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Optimal solution:  $x = 2.5$ ,  
 $y = 2.5$

Solution value =  $7.5 + 5 =$   
 $12.5$

Half paintings?



# Integer (linear) program

maximize  $3x + 2y$

subject to

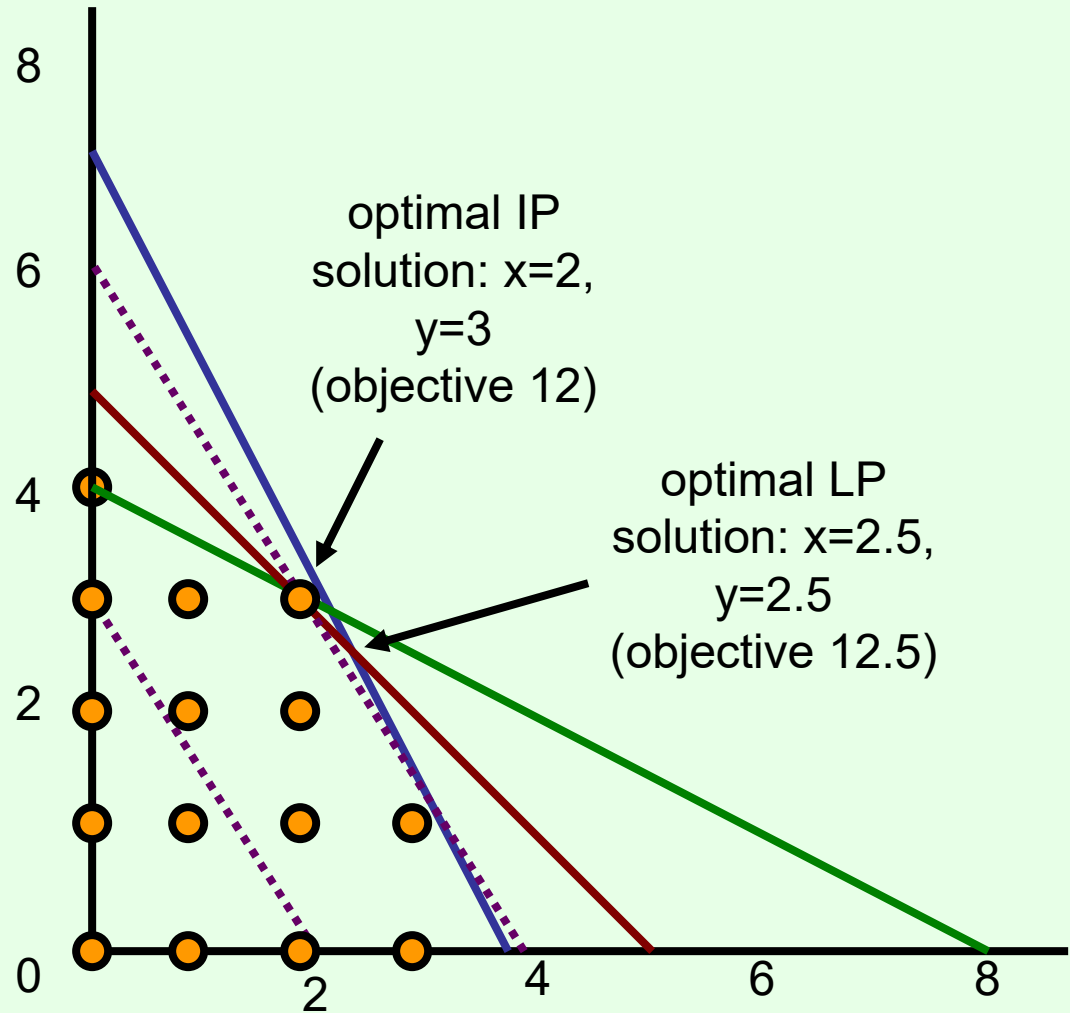
$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$x \geq 0$ , integer

$y \geq 0$ , integer



# Mixed integer (linear) program

maximize  $3x + 2y$

subject to

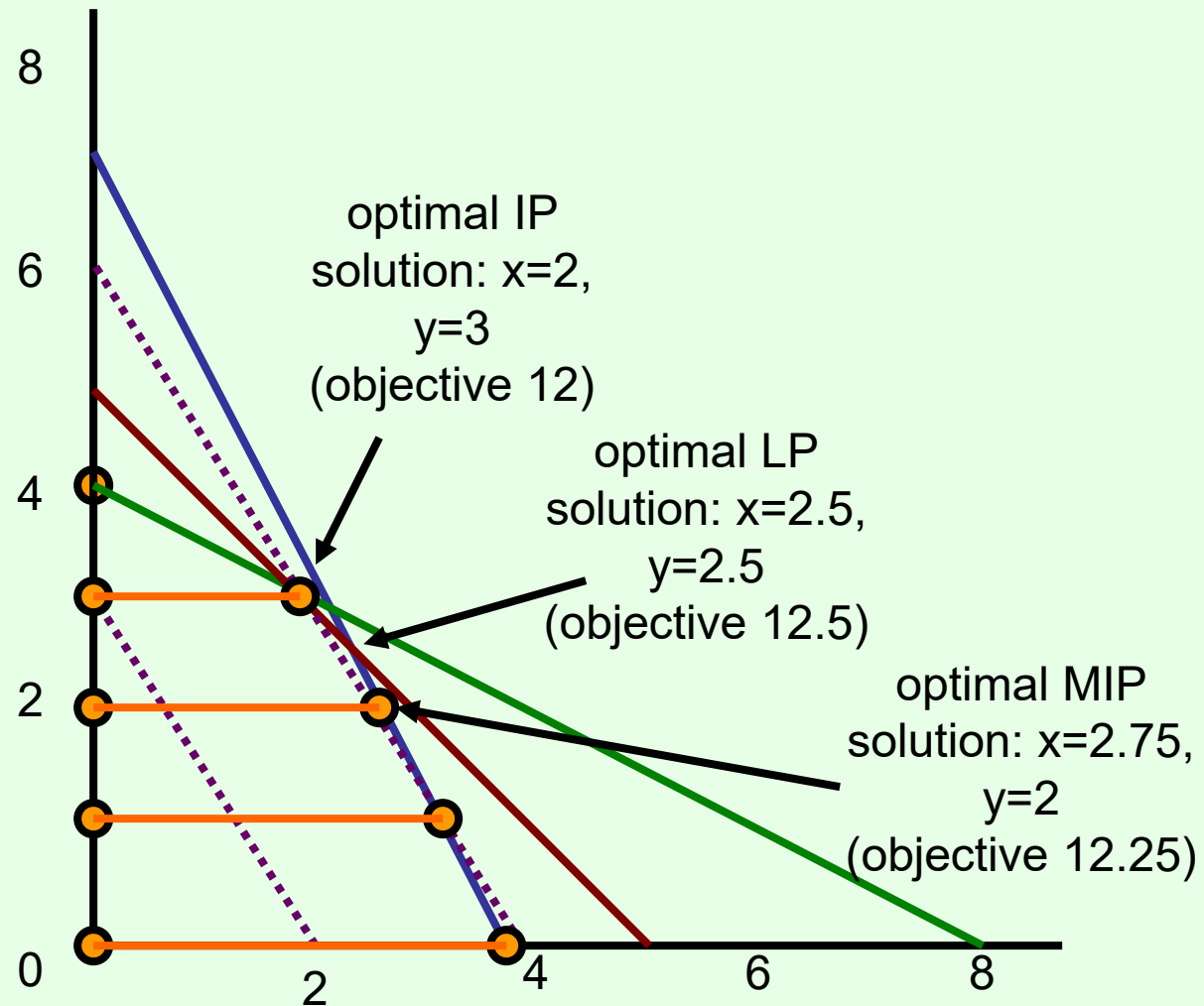
$$4x + 2y \leq 15$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0, \text{ integer}$$



# Exercise in modeling: knapsack-type problem

- We arrive in a room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for \$11
  - There are 3 units available
- Unit of object B: 4kg, 4 liters, sells for \$4
  - There are 4 units available
- Unit of object C: 6kg, 3 liters, sells for \$9
  - Only 1 unit available
- What should we take?

# The *general* version of this knapsack IP

maximize  $\sum_j p_j x_j$

subject to

$$\sum_j w_j x_j \leq W$$

$$\sum_j v_j x_j \leq V$$

$$(\text{for all } j) x_j \leq a_j$$

$$(\text{for all } j) x_j \geq 0, x_j \text{ integer}$$

# Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- ... but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

# Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)

