COMPSCI 330: Design and Analysis of Algorithms	02/18/2016 and 02/23/2016
Network Flows	
Lecturer: Debmalya Panigrahi	*Scribe: Tianqi Song*

1 Overview

This lecture covers the basic definitions and properties of network flows. The maximum flow problem is also discussed.[1]

2 Preliminaries

2.1 Network

A network is a directed graph $G = (V, E)$ where there exists a source vertex $s \in V$ and a sink vertex $t \in V$. Each edge $(v, w) \in E$ has a positive capacity $u(v, w)$. An example of network is shown in Figure 1

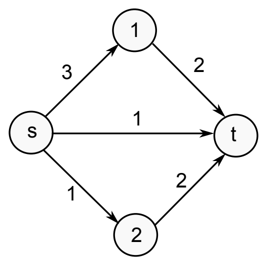

Figure 1: An example of network.

2.2 Raw Flow and Net Flow

A raw flow is a function $r(v, w) \geq 0$, where $(v, w) \in E$, that satisfies the following properties:

$$Capacity: r(v,w) \leq u(v,w),\ \forall (v,w) \in E \tag{1}$$
$$Balance: \Sigma_{w \in V} r(w,v) = \Sigma_{w \in V} r(v,w),\ \forall v \in V \setminus \{s,t\} \tag{2}$$

A net flow is a function $f(v, w)$, where $(v, w) \in V^2$, that satisfies the following properties:

$$Skew\ symmetry: f(v,w) = -f(w,v) \tag{3}$$
$$Capacity: f(v,w) \leq u(v,w),\ \forall (v,w) \in V^2 \tag{4}$$
$$Balance: \Sigma_{w \in V} f(v,w) = 0,\ \forall v \in V \setminus \{s,t\} \tag{5}$$

In this note, we use "net flow" for all the definitions and proofs.

[1]Some materials are from previous notes by Ang Li and Wenshun Liu for this class in Fall 2014.

2.3 Value of a Flow

$$Value\ of\ a\ flow\ f:\ |f| = \Sigma_v f(s,v) = \Sigma_v f(v,t) \tag{6}$$

2.4 Maximum Flow Problem

Find the flow of maximum value on a given network.

2.5 Residual Network

Given a network $G = (V,E)$ and a flow f, we can construct a residual network G_f with capacity function $u_f(v,w) = u(v,w) - f(v,w)$.

Lemma 1. *If f' is a flow on a residual network G_f, $f'' = f + f'$ is a flow on G.*

Proof. Skew symmetry: $f''(v,w) = f(v,w) + f'(v,w) = -f(w,v) - f'(w,v) = -f''(w,v)$. Capacity: $f'(v,w) \leq u(v,w) - f(v,w) \implies f''(v,w) = f'(v,w) + f(v,w) \leq u(v,w)$. Balance: $\forall v \in V \setminus \{s,t\}$, $\Sigma_w f(v,w) = 0$ and $\Sigma_w f'(v,w) = 0 \implies \Sigma_w f''(v,w) = \Sigma_w f(v,w) + \Sigma_w f'(v,w) = 0$. □

2.6 Augmenting Path

An augmenting path is a path from source s to sink t in a residual network.

2.7 Cut and s-t Cut

A cut is a partition of vertices into two sets S and T, where $S = V \setminus T$. An s-t cut is a cut such that $s \in S$ and $t \in T$.

2.8 Capacity of a Cut

We define the capacity of a cut S and T as $u(S) = \Sigma_{(v,w) \in S \times T} u(v,w)$.

Lemma 2. *The value of any flow f on G is no more than the capacity of any s-t cut S, T.*

Proof. $|f| = \Sigma_{v \in V} f(s,v) = \Sigma_{v \in V} f(s,v) + 0 = \Sigma_{v \in V} f(s,v) + \Sigma_{v \neq s, v \in S} \Sigma_{w \in V} f(v,w) = \Sigma_{v \in S} \Sigma_{w \in V} f(v,w)$
$= \Sigma_{(v,w) \in S \times T} f(v,w) + \Sigma_{(v,w) \in S \times S} f(v,w) = \Sigma_{(v,w) \in S \times T} f(v,w) \leq \Sigma_{(v,w) \in S \times T} u(v,w)$. □

3 Augmenting Path Algorithm

The pseudocode is:

Algorithm 1 Augmenting Path Algorithm

```
function AUGMENT(G)
    G_f = G
    f(v,w) = 0 ∀(v,w) ∈ E
    while ∃ s-t path p in G_f do
        f' = flow on path p
        f = f + f'
        Recompute G_f
```

Lemma 3. *The value of flow f computed by this augmenting path algorithm is equal to the capacity of an s-t cut S, T, where S is the set of vertices that are reachable from s in G_f.*

Proof. By the definition of S, there is no edge from S to T in G_f. Therefore, (1) the edges from S to T in G are saturated by f and (2) the edges from T to S in G are empty under f. Otherwise, there will be contradiction. □

By lemma 1, 2 , 3, we have:

Theorem 4. *The augmenting path algorithm computes the maximum flow and the value of maximum flow is equal to the capacity of minimum cut.*

The time complexity of this algorithm is $O(|E| f_{max})$.