Due on April ~~16~~ 18, 2018
85 points total

General Directions: If you are asked to provide an algorithm, you should clearly define each step of the procedure, establish its correctness, and then analyze its overall running time. There is no need to write pseudo-code; an unambiguous description of your algorithm in plain text will suffice.

All the answers must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a pdf file will also be accepted. Before submitting the pdf file, please make sure that it can be opened using any standard pdf reader (such as Acrobat Reader) and your entire answer is readable. **Handwritten answers or pdf files that cannot be opened will not be graded and will not receive any credit.**

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Problem 1 (25 points)
Your old friend Bob Bitfiddler is looking to open two new convenience stores in the small town of Linesville, NC. Linesville is composed of a single east-west street called Straight Street. There are n homes in total along Straight Street, and Bob has surveyed the residents of each of them in order to optimize the locations of his stores. For each house i located $\ell_i \in [0, 10]$ miles from the west gate of the town, his survey indicates there are r_i residents in home i, all of whom are willing to walk at most $d_i \in [0, 10]$ miles to shop at one of Bob's two new stores. Bob's goal is to pick the locations of his two stores $s_1, s_2 \in [0, 10]$ along Straight Street that maximize the total number of residents that are willing to visit at least one of the stores.

Formulate the above problem as an integer programming problem (i.e., the optimal solution to your IP should solve for the two locations that maximize the total number of shoppers). Also provide a brief justification for why your formulation is correct.

Some notes:

- Although s_1 and s_2 could be fractional locations (i.e, they need not be placed at mile $1, 2, \ldots$), you can still restrict other variables to be integers since we are asking for an integer programming formulation.
- As an intermediate step, use absolute values in your formulation. Then, try to rewrite the IP without absolute values.

Problem 2 (25 points)
This problem is about the minimum spanning tree (MST) problem in an undirected graph, where every edge has a non-negative cost.

(a) (15 points) Write an LP for the MST problem, which enforces the constraint that at least one edge is selected from every cut. Show that you can solve this LP in polynomial time by using an efficient separation oracle. (It is sufficient to identify the problem you need to solve in the separation oracle; you do not need to give an explicit algorithm.) Give an example to show that this LP is not integral.
(Recall that an LP is said to be integral if there is at least one optimal solution where all the variables take integer values.)

(b) (10 points) Write a different LP for the MST problem where you enforce an upper bound on the number of edges selected from any induced subgraph, and a lower bound on the total number of edges. Show that any integral feasible solution to this LP is a spanning tree of the graph.

Problem 3 (35 points)
Consider the following problem: given an undirected graph $G = (V, E)$, find a smallest subset of vertices S with the property that for every vertex $v \in V$, $|N_v \cap S| \geq 1$, where $N_v = \{v\} \cup \{u : (u, v) \in E\}$.

(a) (5 points) Formulate the above problem using an integer linear program (ILP).

(b) (10 points) Give the fractional linear program corresponding to the above ILP and show that it is not integral, i.e., give an example demonstrating an integrality gap greater than 1 for this LP.

(c) (5 points) Write the dual of this LP.

(d) (5 points) Write the ILP corresponding to this dual LP, and state the problem it represents in words.

(e) (10 points) Show that this dual LP is also not integral, i.e., give an example demonstrating an integrality gap greater than 1 for the dual LP.