COMPSCI 230: Discrete Mathematics for Computer Science	February 13, 2019
Lecture 10	
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1 Overview

In this lecture, we introduce a common class of relations: partial orders. We define both weak and strict partial orders, and overview their properties. We conclude with Dilworth's Theorem.

2 Partial Orders

Definition 1. *A relation R is a* weak partial order *(or simply a* partial order*) if and only if R is transitive and anti-symmetric.*

In contrast to weak partial orders, we also define a stricter variant known as strict partial orders.

Definition 2. *A relation R is a* strict partial order *if and only if R is transitive and asymmetric.*

Since a strict partial order is asymmetric, it is also irreflexive, whereas a weak partial order may or may not be reflexive or irreflexive. Consider the following examples.

Example 1: Let S be some non-empty set, and 2^S be the power set of S. The relation defined by $\subseteq$ on 2^S is a weak partial order. $\subseteq$ is indeed transitive and anti-symmetric. Note, $\subseteq$ is not a strict partial order since $A \subseteq A$ for any set A. Thus, it is anti-symmetric, but not asymmetric. Instead, the relation defined by a strict subset ($\subset$ on 2^S) is a strict partial order because it is asymmetric and transitive.

Example 2: Suppose $S = \{1, 2, 3\}$. Let R be the relation defined on the power set of S by a strict subset. We can represent this relation with the diagram in Figure 1 where a directed arrow between a and b means a is related to b. Notice that not all arrows of the relation are drawn. If we take the pairs represented by the arrows and apply transitivity, this will yield the entire relation. For instance, there is an arrow between $\emptyset$ and $\{1\}$ and between $\{1\}$ and $\{1, 2\}$. In other words, $(\emptyset, \{1\})$, $(\{1\}, \{1, 2\}) \in R$. Since we know the relation is transitive, we can conclude that $\emptyset$ is related to $\{1, 2\}$. Such a diagram is an intuitive tool for studying partial orders. Notice that no pair has an arrow which goes both ways. This is because the relation is anti-symmetric.

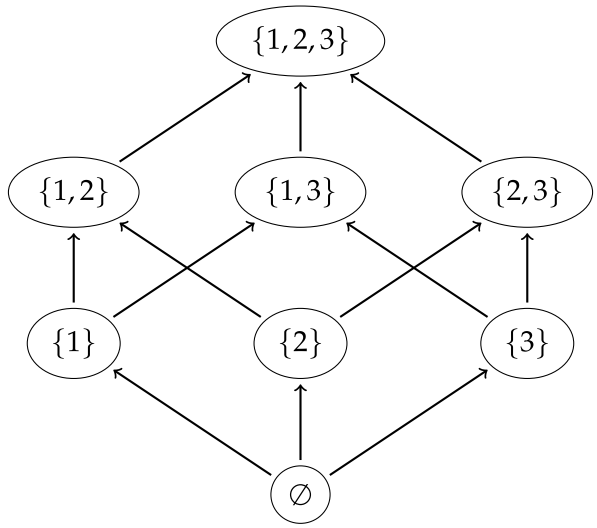

Figure 1: Figure representing relation in Example 2.

Definition 3. *Suppose R is a partial order. Elements $a \in R$ and $b \in R$ are said to be* comparable *if aRb or bRa. (Note that both cannot be true, unless $a = b$). Otherwise, a and b are said to be* incomparable*.*

In Example 2, elements $\{1\}$ and $\{2\}$ are incomparable. On the other hand, elements $\{1,2\}$ and $\{1,2,3\}$ are comparable since $(\{1,2\},\{1,2,3\}) \in R$.

Definition 4. *A set of mutually comparable elements (every two distinct elements of the set are comparable) is called a* chain*. Similarly, a set of mutually incomparable elements (every two distinct elements of the set are incomparable) is called an* anti-chain*.*

Example 3: Let's work with the same partial order defined in Example 2. R is defined by $\subset$ on 2^S for $S = \{1,2,3\}$. The set $\{\emptyset, \{1\}, \{1,2\}\}$ is a chain. Likewise, the set $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3,4\}\}$ is a chain. In contrast, the set $\{\{1\}, \{2\}, \{3\}\}$ is an anti-chain. We can also create a set from the elements of a relation which is neither a chain nor an anti-chain: take $\{\{1\}, \{2\}, \emptyset\}$. Figure 2 shows an example of a chain (in blue) and anti-chain (in red) in the diagram of the relation.

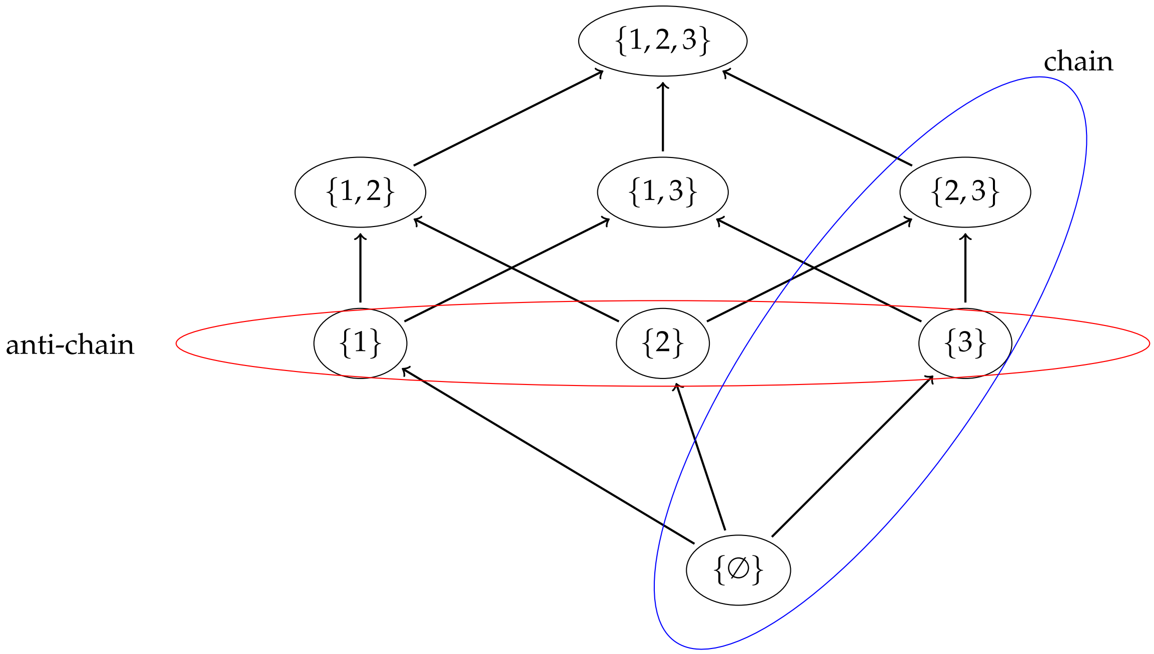

Figure 2: Figure representing relation in Example 3.

Consider our diagram in Figure 1 and Figure 2 again. If we just look at a chain in our diagram it will look like a path where each element is related to the next (i.e. visually, a chain in mathematics looks like a chain in the real world), see Figure 3. In contrast, an anti-chain will have no arrows between any elements, see Figure 4.

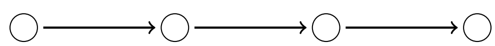

Figure 3: Figure representing a chain.

Figure 4: Figure representing an anti-chain.

Any partial order can be decomposed into chains. In general, this is not a unique decomposition. The trivial decomposition has chains consisting of a single element. A set with only a single element of a partial order is both a chain and an anti-chain, as there are no pairs that could be related.

Example 4: We will continue to use the partial order from Example 2 and 3. R is defined by $\subset$ on 2^S for $S = \{1,2,3\}$. The trivial decomposition of R into chains consists of $2^{|S|} = 8$ chains. Another possible decomposition:

$$
\begin{aligned}
C_1 &= \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\} \\
C_2 &= \{\{2\}, \{2,3\}\} \\
C_3 &= \{\{3\}, \{1,3\}\}
\end{aligned}
$$

The decomposition into chains form a partition of the elements of a relation. In this example we see that the trivial decomposition consists of 8 chains, and we found another decomposition consisting of only 3 chains. Let's consider another example using a different relation.

Example 5: Define a partial order on $\mathbb{Z}^+$ using the relation defined by $<$. In a previous lecture, we verified that $<$ is both transitive and anti-symmetric. We can draw the diagram for this partial order, see Figure 5. We omit the arrows we would obtain by applying the transitive property to the related pairs shown.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \rightarrow \cdots$$

Figure 5: Figure representing relation in Example 4.

We can decompose this partial order into one chain, in fact it is exactly the chain depicted in Figure 5. This leads to the following definition.

Definition 5. *A partial order that is a single chain is called a* total order *or a* linear order*.*

Often, the integers are said to be 'totally ordered.' This really means that we can define a relation on the integers which is a total order. Consider constructing an anti-chain for the partial order in Example 5. For any two distinct positive integers a and b, either $a < b$ or $b < a$. Thus, the largest anti-chain we could create only has one element. A curious fact is that in both examples 3 and 4, we have found a decomposition of the partial order into chains where the number of chains is equal to the size of an antichain in the partial order. This is not a coincidence, as shown in a theorem published by the mathematician Robert Dilworth in 1950.

Theorem 1 (Dilworth, 1950)**.** *In any partial order, the minimum number of chains in any partition of the relation into chains is exactly equal to the largest anti-chain.*

Consider some partial order P. To prove Theorem 1, we can show that the minimum number of chains P can be decomposed into is at lease the maximum size of an anti-chain in P *and vice-versa*. Showing the first direction is straightforward.

(One direction of the) Proof of Dilworth's Theorem. Let P be some partial order. Suppose the largest anti-chain of P is of size n. By definition of an anti-chain, any two of these n elements are incomparable, and thus must be in distinct chains in any decomposition of P into chains. Therefore, the minimum number of chains P can be decomposed into is at least n, the maximum size of an anti-chain of P.

How do we prove the other direction? We will need a proof technique known as mathematical induction. We will come back to finish the proof of this theorem after studying this technique in the next lecture. □

3 Summary

In this lecture, we introduced partial orders, a useful categorization of relations. We defined comparable and incomparable elements, and introduced chains and anti-chains. We finally concluded with Dilworth's theorem, regarding the relationship between the largest anti-chain and smallest size decomposition of a partial order.