

Lecture 11

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1 Overview

In this lecture, we study mathematical induction, which we often use to prove that every nonnegative integer satisfies some given property. Induction comes in two forms—weak and strong—and we will see their differences, as well as examples of both.

2 Weak Induction

We begin by studying weak (also known as ordinary) induction. Suppose we want to prove that some predicate $P(n)$ is true for every number in \mathbb{Z}_0^+ , the set of nonnegative integers. We first state the driving principle behind weak induction, which is that the following implication is true:

$$[P(0) \wedge (\forall n. P(n) \rightarrow P(n+1))] \rightarrow [\forall n. P(n)],$$

where the universe of n is \mathbb{Z}_0^+ . Notice that the consequence (the part after the second “ \rightarrow ”) of the above implication is what we want to prove. So if we can prove the antecedent (the part before the second “ \rightarrow ”), then since the entire implication is true, we would be done. Proving the antecedent requires proving two things: $P(0)$ and, for every $n \in \mathbb{Z}_0^+$, $P(n) \rightarrow P(n+1)$. The first proposition is known as the “base case,” and the second is known as the “inductive step.”

Now why is the implication true? If the antecedent is true, then $P(0)$ is true, and furthermore, $P(0) \rightarrow P(1)$ is true, so $P(1)$ is true. Continuing, since $P(1)$ and $P(1) \rightarrow P(2)$ are true, we can conclude $P(2)$ is also true. Thus, through this chain of implications, we can conclude $P(n)$ is true for every $n \in \mathbb{Z}_0^+$, as desired. We now illustrate weak induction with an example.

Theorem 1. For any $n \in \mathbb{Z}_0^+$, the following holds:

$$\sum_{i=0}^n i = \frac{n(n+1)}{2}.$$

In this theorem, we have $P(n) = “\sum_{i=0}^n i = n(n+1)/2”$, so the theorem states

$$\forall n \in \mathbb{Z}_0^+. P(n).$$

We shall now prove Theorem 1 using ordinary induction.

Proof. Recall that in ordinary induction, we must prove the following things:

$$P(0) \text{ and } \forall n \in \mathbb{Z}_0^+. P(n) \rightarrow P(n+1).$$

Base case: $P(0)$ is true because both sides of the equality in $P(0)$ are equal to 0:

$$\sum_{i=0}^0 i = 0 = \frac{0(0+1)}{2}.$$

Inductive Step: We now show $P(n) \rightarrow P(n+1)$ for every $n \in \mathbb{Z}_0^+$. Let n be any nonnegative integer, and assume $P(n)$ is true. This assumption is known as the **inductive hypothesis**. In our case, assuming $P(n)$ is true is equivalent to assuming

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

is true. We want to use this inductive hypothesis to prove $P(n+1)$, which is the following:

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}.$$

The left side of $P(n+1)$ can be written as follows:

$$\begin{aligned} \sum_{i=1}^{n+1} i &= 1 + 2 + \cdots + (n-1) + n + (n+1) \\ &= \sum_{i=1}^n i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) && \text{(by the inductive hypothesis)} \\ &= (n+1) \left(\frac{n}{2} + 1 \right) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This final expression is the right side of $P(n+1)$. Thus, by assuming $P(n)$ is true, we have shown that $P(n+1)$ is true. By the principle of induction, we can conclude that $P(n)$ is true for all $n \in \mathbb{Z}_0^+$. \square

To summarize, a proof by weak induction that proves a predicate $P(n)$ for $n \in \mathbb{Z}_0^+$ has the following steps:

1. **Base Case:** Prove that $P(0)$ is true.
2. **Inductive Hypothesis:** Precisely state the hypothesis that $P(n)$ is true.
3. **Inductive Step:** Prove that $P(n+1)$ is true using the inductive hypothesis.

Now let's see an example of induction being used incorrectly.

Claim 2 (Incorrect). *In any set of n nonnegative integers, all of the elements are equal to each other.*

“Proof” of Claim 2: Here, the predicate $P(n)$ is simply the statement of the claim.

Base Case: A set with 0 elements contains no elements, so they are all equal to each other, so $P(0)$ is true.

Inductive Hypothesis: In any set of n natural numbers, all elements of a set are equal.

Inductive Step: We want to show that if S is a set containing $n + 1$ nonnegative integers, then all elements of S are equal to each other. Let a and b be two distinct (different) elements of S , and consider the subsets $S_1 = S \setminus \{a\}$ and $S_2 = S \setminus \{b\}$.

Since S_1 and S_2 are sets of size n , we can apply our inductive hypothesis on each and conclude that all elements of S_1 are equal to each other, and all elements of S_2 are equal to each other. Now let c be any element in $S_1 \cap S_2$. Since a and c are both in S_1 , the induction hypothesis tells us $a = c$. Similarly, b and c are both in S_2 , so $b = c$. Thus, $a = b = c$ and in fact, every element of $S_1 \cup S_2$ is equal to c as well. Thus, every element of $S = S_1 \cup \{a\}$ is equal to the value of c . \square

What went wrong? We assumed the existence of c . If $n = 1$, then $S_1 = \emptyset$, so we cannot find a c that's in both S_1 and S_2 . In fact, when $n = 1$, we cannot even find two distinct elements of S . Thus, we did not prove the proposition $P(0) \rightarrow P(1)$, so we have not met the conditions for ordinary induction.

3 Strong Induction

Now we will introduce a more general version of induction known as *strong induction*. The driving principle behind strong induction is the following proposition which is quite similar to that behind weak induction:

$$[P(0) \wedge (\forall n. (P(0) \wedge P(1) \wedge \cdots \wedge P(n)) \rightarrow P(n+1))] \rightarrow [\forall n. P(n)],$$

Again, the universe of n is \mathbb{Z}_0^+ . Notice that this is similar to weak induction: the conclusion is the same, we still have $P(0)$ as a base case, and the second part of the antecedent is also an implication. This time, however, the antecedent of the implication is

$$P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n).$$

Recall that in weak induction, the antecedent was only $P(n)$. This means that strong induction allows us to assume n predicates are true, rather than just 1, when proving $P(n + 1)$ is true.

For example, in ordinary induction, we must prove $P(3)$ is true assuming $P(2)$ is true. But in strong induction, we must prove $P(3)$ is true assuming $P(1)$ and $P(2)$ are *both* true. Note that any proof by weak induction is also a proof by strong induction—it just doesn't make use of the remaining $n - 1$ assumptions. We now proceed with examples.

Recall that a positive integer has a *prime factorization* if it can be expressed as the product of prime numbers.

Theorem 3. *Any positive integer greater than 1 has a prime factorization.*

Proof. Here, the predicate we want to prove is $P(n) = “n \text{ has a prime factorization}.”$

Base case: Note that the base case is $n = 2$, because the claim does not apply when $n \in \{0, 1\}$. Since 2 itself is prime, we can conclude $P(2)$ is true.

Inductive hypothesis: Every integer between 2 and n (including 2 and n) has a prime factorization.

Inductive step: Let $n + 1$ be any positive integer greater than 1. If $n + 1$ is prime, then $n + 1$ itself is a prime factorization of n . Otherwise, $n + 1 = a \cdot b$ where $a, b \in \{2, \dots, n\}$. Since a and b are between 2 and n , the inductive hypothesis tells us that a and b each have a prime factorization, so they can be expressed as the product of prime numbers. Since $n + 1$ is the product of a and b , this implies $n + 1$ can also be expressed as the product of prime numbers, so $P(n + 1)$ is true. \square

Note: It would be much more difficult to prove Theorem 3 using weak induction, because we don't know how small a and b might be. With weak induction, we would only know n has a prime factorization, but it is entirely possible (and always true) that a and b are less than n .

Now let's consider a second example. Suppose all of the students in COMPSCI 230 are participating in a tournament. Each game is played between two players (e.g., Tic-tac-toe). The tournament is structured as follows: first, the instructor splits all of the students into two groups, called Group 1 and Group 2. (The size of the groups can be anything greater than zero.) The instructor then has everybody in Group 1 play everybody in Group 2 exactly once. Then, the instructor splits Group 1 into two groups (again, with arbitrary sizes), and every student of each group plays every student of the other group. The instructor does the same for Group 2. This process continues until every group has one student.

For example, suppose there are 6 students and the instructor creates groups of size 2 and 4. This creates $2 \cdot 4 = 8$ games. Then the group of 2 gets split into 1 and 1, creating 1 game. The group of 4 gets split into 1 and 3, creating 3 games. This group of 3 then gets split into 1 and 2, creating 2 games. Finally, this group of 2 gets split into 1 and 1, creating 1 game. Thus, the total number of games played is $8 + 1 + 3 + 2 + 1 = 15$.

If the initial split was size 3 and 3, then we would obtain a different list of numbers in the final sum. However, the surprising result is the following: no matter how the instructor chooses to split each group, the total number of games created remains the same!

In fact, let us determine the total number of games in terms of n . If x and y are two students, then the moment x and y get split, they play each other once. Afterwards, they never play each other again, because from then on, they only play students within their current group. Thus, the total number of games is $\binom{n}{2} = n(n - 1)/2$. (Notice that in the example above, we indeed have $6(6 - 1)/2 = 15$ total games.) We will now prove this fact using strong induction.

Theorem 4. *If the splitting procedure described above is applied to a class of n students, then the total number of games played is always $n(n - 1)/2$, regardless of the sizes of the groups.*

Proof. We proceed by strong induction.

Base case: The instructor never forms a group of size 0, so the base case is $n = 1$. If there's only one student, then the total number of games played is 0, and $1(1 - 1)/2$ is indeed 0.

Inductive hypothesis: For any $x \leq n$, the total number of games that x students play (via any splitting procedure) is $x(x - 1)/2$. Note that we will assume $P(1) \wedge \dots \wedge P(n)$ and prove $P(n + 1)$.

Inductive step: Suppose the total number of students is $n + 1$, and the first split contains a group of size y and a group of size $n + 1 - y$. This results in $y(n + 1 - y)$ games. Now we have two separate groups, and these groups never interact in the future. Thus, we can apply the inductive hypothesis on each group and conclude that the first creates $y(y - 1)/2$ games while the second

creates $(n + 1 - y)(n - y)/2$ games. Thus, the total number of games is

$$\begin{aligned}
 & y(n + 1 - y) + \frac{y(y - 1)}{2} + \frac{(n + 1 - y)(n - y)}{2} \\
 &= ny + y - y^2 + \frac{y^2 - y}{2} + \frac{n^2 + n - 2ny - y + y^2}{2} \\
 &= \frac{2ny + 2y - 2y^2 + y^2 - y + n^2 + n - 2ny - y + y^2}{2} \\
 &= \frac{n^2 + n}{2} = \frac{n(n + 1)}{2}.
 \end{aligned}$$

□

Note: Again, proving Theorem 4 using ordinary induction would be difficult, because in order to apply the inductive hypothesis, we would have to assume that the first split creates groups of size 1 and n , which may not be true.

Finally, let's look at the sequence of Fibonacci numbers, named after the Italian mathematician Fibonacci. The sequence begins with 0 and 1, and then each term is the sum of the previous two terms. More formally, the n -th Fibonacci number $F(n)$ is defined as follows:

$$\begin{aligned}
 F(0) &= 0 \\
 F(1) &= 1 \\
 F(n) &= F(n - 1) + F(n - 2) \quad \text{if } n \geq 2.
 \end{aligned}$$

Theorem 5. Every third Fibonacci number is even, that is for all $n \geq 0$,

$$F(n) \text{ is even} \leftrightarrow F(n + 3) \text{ is even}.$$

Notice that the statement of Theorem 5 already suggests using strong induction. Since $F(n)$ is defined by the previous *two* terms (not one), proving Theorem 5 using ordinary induction would be much more difficult. However, as we shall see, this proof does not make full use of the inductive hypothesis (unlike the proofs of Theorem 3 and 4).

Proof. We proceed by strong induction.

Base case: Here, the base case involves both $n = 0$ and $n = 1$ because of the way that Fibonacci numbers are defined. For $n = 0$, we see that $F(0) = 0$ and $F(0 + 3) = 2$ are both even. For $n = 1$, we see that $F(1) = 1$ and $F(4) = 3$ are both odd. In either case, we've proven the predicate

$$P(n) = "F(n) \text{ is even} \leftrightarrow F(n + 3) \text{ is even}."$$

Inductive hypothesis: Assume $P(1), P(2), \dots, P(n)$ are true. In particular, this means $P(n - 1)$ and $P(n)$ are both true.

Inductive step: Recall that if a and b are integers, then

$$a + b \text{ is even} \leftrightarrow [a \text{ is even} \leftrightarrow b \text{ is even}]. \quad (1)$$

(It is straightforward to prove this by cases.) Now consider $F(n + 1)$ and notice the following:

$$\begin{aligned}
 F(n + 1) \text{ is even} &\leftrightarrow F(n) + F(n - 1) \text{ is even} && \text{(definition of } F(n + 1)) \\
 &\leftrightarrow [F(n) \text{ is even} \leftrightarrow F(n - 1) \text{ is even}] && \text{(from (1) above)} \\
 &\leftrightarrow [F(n + 3) \text{ is even} \leftrightarrow F(n + 2) \text{ is even}],
 \end{aligned}$$

where the final equivalence holds by the inductive hypothesis because we have assumed $P(n)$ and $P(n - 1)$ are both true. Continuing, the above is equivalent to

$$\begin{aligned} F(n + 3) + F(n + 2) \text{ is even} & \qquad \qquad \qquad \text{(from (1) above)} \\ \Leftrightarrow F(n + 4) \text{ is even.} & \qquad \qquad \qquad \text{(definition of } F(n + 4)) \end{aligned}$$

Thus, we have shown that $F(n + 1)$ is even if and only if $F(n + 4)$ is even, which is exactly the statement $P(n + 1)$. \square

4 Summary

In this lecture, we studied both the weak and the strong versions of mathematical induction. We also looked at their differences, saw how strong induction is easier to use in many situations than weak induction, and proved multiple theorems using induction.