COMPSCI 230: Discrete Mathematics for Computer Science	February 27, 2019
Lecture 13	
Lecturer: Debmalya Panigrahi	*Scribe: Erin Taylor*

1 Overview

In this lecture, we begin our study of graph theory. We introduce common classes of graphs and prove some simple properties of graphs. Finally, we study matchings in bipartite graphs.

2 Basic Definitions for Graphs

A graph $G = (V, E)$ is an ordered pair of sets. V denotes the set of vertices, and E the set of edges. For our purposes, V is a finite, non-empty set. $E \subseteq V^{(2)}$, where $V^{(2)}$ is the set of 2-element subsets of V. It is common notation to let $|V| = n$ and $|E| = m$. Let's consider an example.

Example 1: See Figure 1.
$V = \{1, 2, 3\}$
$V^{(2)} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
$E = \{\{1, 2\}, \{1, 3\}\}$

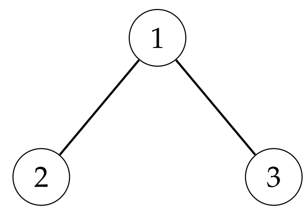

Figure 1: Graph in Example 1

We say that the two vertices an edge connect are the 'endpoints' of the edge. The edge is said to be incident on its two endpoints. Graphs defined as above are said to be undirected, and can be used to represent symmetric relationships. For example an undirected graph could represent the relations 'is a friend of' or 'was born in the same month as'.

Definition 1. *The* degree *of a vertex $v \in V$, denoted $d(v)$, is the number of edges incident on v.*

We have the following simple lemma characterizing the relationship between the total sum of degrees in a graph and the number of edges.

Lemma 1. *The sum of degrees of vertices in a graph is twice the number of edges, i.e.,*

$$\sum_{v \in V} d(v) = 2m.$$

Proof. We will prove Lemma 1 by induction. We will induct on the number of edges in the graph. For non-negative m, we will prove the lemma holds for any graph with m edges.
Base Case: If a graph $G = (V, E)$ has no edges ($E = \emptyset$), then every vertex has degree 0. Thus, $\sum_{v \in V} d(v) = 0 = 2(0)$.
Inductive Case: For our inductive hypothesis, we assume the claim holds for any graph with less than m edges. Then, we want to prove the claim for an arbitrary graph $G = (V, E)$ where $|E| = m$.

Fix an arbitrary edge of G and remove it. Let this edge be $e = \{a, b\}$. Let $G' = (V, E')$ where $E' = E \setminus \{e\}$. Then, G' is a graph with $m - 1$ edges. Let $d'(v)$ be the degree of $v \in V$ in G'. By our inductive hypothesis,

$$\sum_{v \in V} d'(v) = 2(m-1).$$

For all $v \in V \setminus \{a, b\}$, $d'(v) = d(v)$. Additionally, $d(a) = 1 + d'(a)$ and $d(b) = 1 + d'(b)$ since a, b are endpoints of edge e, and so are incident on one more edge in G than in G'. Thus,

$$\sum_{v \in V} d(v) = 2 + \sum_{v \in V} d'(v) = 2 + 2(m-1) = 2m.$$

Therefore, for any graph $G = (V, E)$ the theorem holds. □

We now introduce the concept of subgraphs. This is similar to the idea of subsets, but for graphs instead of sets.

Definition 2. *A* subgraph $G' = (V', E')$ *of graph* $G = (V, E)$ *satisfies the property that* $V' \subseteq V$ *and* $E' \subseteq E$.

See Figure 2 and Figure 3 for an example of a subgraph.

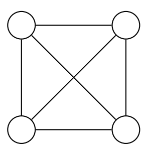

Figure 2: A graph

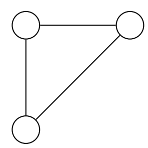

Figure 3: A subgraph of the graph in Figure 2.

3 Special Types of Graphs

Many categories of graphs arise frequently in both practice and theory, and have special names.

- **Empty Graph**: The empty graph is a graph with no edges, i.e. $E = \emptyset$. See Figure 4 for an example.
- **Complete Graph**: The complete graph is a graph with all possible edges, i.e. $E = V^{(2)}$. See Figure 5 for an example.
- **Cycle Graph**: A cycle graph must have at least three vertices. Suppose graph G had vertices $V = \{1, 2, \ldots, n\}$. Then G is a cycle graph if $E = \{\{1,2\}, \{2,3\}, \ldots \{n-1, n\}, \{n, 1\}\}$. Notice that $|E| = n$. See Figure 6 for an example.

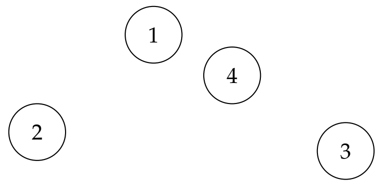

Figure 4: An empty graph with 4 vertices.

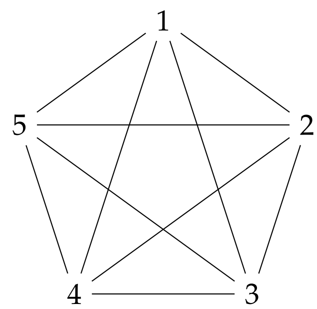

Figure 5: A complete graph with 5 vertices

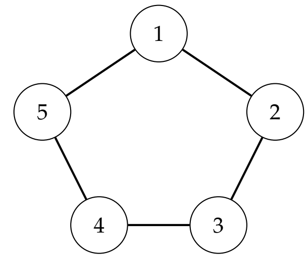

Figure 6: An cycle graph with 5 vertices.

Lemma 2. *An n-vertex complete graph has $\frac{n(n-1)}{2}$ edges.*

Proof. We will prove Lemma 2 using induction. We will induct on the number of vertices in the graph. For positive n, we will prove the lemma holds for any graph with n vertices.
Base Case: If a graph a complete $G = (V, V^{(2)})$ has 1 vertex, then G has no edges and $n(n-1)2 = 1(0)/2 = 0$.
Inductive Case: For our inductive hypothesis, we assume a complete graph with $1 \leq n' < n$ vertices has $n'(n'-1)/2$ edges. Now, we will prove the lemma for an arbitrary n-vertex complete graph. Fix a vertex, $v \in V$. Create G' by removing v and all edges incident on v. G' is a complete graph with $n-1$ vertices. Thus, by our inductive hypothesis, G' has $\frac{(n-1)(n-2)}{2}$ edges. The number of edges in G is the number of edges in G' and all edges incident on v. There are exactly $n-1$ edges incident to v. Therefore:

$$|E| = (n-1) + \frac{(n-1)(n-2)}{2} = (n-1)\left(1 + \frac{n-2}{2}\right) = \frac{n(n-1)}{2}.$$

Thus, any n-vertex complete graph has $\frac{n(n-1)}{2}$ edges. □

Another graph that arises frequently in practice are bipartite graphs.

Definition 3. *A* bipartite *graph is a graph $G = (V, E)$ in which V can be partitioned into two sets A, B such that $E \subseteq \{\{x, y\} : x \in A, y \in B\}$. Thus, $V = A \cup B$ and $A \cap B = \emptyset$.*

Bipartite graphs arise frequently in practice. For example, suppose we have a group of mentors and mentees. We can model this situation using a bipartite graph. Edges represent the willingness of a pair to have a mentor/mentee relationship. There will be no edges within the set of mentors or within the set of mentees, since a mentor does not need to be mentored and a mentee cannot be a mentor. We have the following property of bipartite graphs.

Lemma 3. *For a bipartite graph $G = (V, E)$ where $V = A \cup B$ and $|E| = m$,*

$$\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m$$

Proof. Let $G = (V, E)$ be an arbitrary bipartite graph where $V = A \cup B$. Each vertex is either in A or B, but not both since A and B are disjoint. Thus,

$$\sum_{v \in V} d(v) = \sum_{v \in A} d(v) + \sum_{v \in B} d(v) = 2m.$$

The last equality holds by Lemma 1. Every edge has one endpoint in A and one endpoint in B, so $\sum_{v \in A} d(v) = \sum_{v \in B} d(v)$. Thus, $2m = 2\sum_{v \in A} d(v)$. This implies

$$\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m. \qquad \square$$

4 Matchings in Bipartite Graphs

Consider again a bipartite graph representing mentors and mentees. We might want to assign a unique mentor to each mentee. To do so, our graph would have to satisfy some additional properties. First, let us introduce some more definitions.

Definition 4. *A* bipartite matching *is a bipartite graph $G = (V, E)$ such that for every vertex $v \in V$, $d(v) \leq 1$.*

See Figure 7 for an example of a bipartite matching.

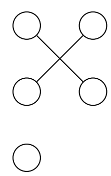

Figure 7: A bipartite matching.

Definition 5. *A* perfect matching *is a bipartite graph $G = (V, E)$ such that for every vertex $v \in V$, $d(v) = 1$.*

Figure 9, 10 show two perfect matchings of the bipartite graph in Figure 8. In the case of matching mentors to mentees, we are asking whether there exists a perfect matching which is a subgraph of the original bipartite graph. A perfect matching may not always exist, see Figure 11. For a perfect matching to exist in graph $G = (V = A \cup B, E)$, it must be that $|A| = |B|$. This is a necessary condition, but not sufficient. A famous result known as Hall's Marriage Theorem gives necessary and sufficient conditions for the existence of a perfect matching in a bipartite graph. To state this theorem precisely, we need to introduce the notion of an A-perfect matching.

Definition 6. *An A-*perfect matching *is a bipartite graph $G = (V, E)$ with $V = A \cup B$ and $E \subseteq \{(u, v) : u \in A, v \in B\}$ such that for every $u \in A$, $d(u) = 1$, and for every $v \in B$, $d(v) \leq 1$.*

We also need the definition of the neighborhood of A set. This is simply the set of vertices that are connected to the vertices of the set by edges in the graph.

Definition 7. *In a bipartite graph $G = (V, E)$ with $V = A \cup B$ and $E \subseteq \{(u, v) : u \in A, v \in B\}$, the* neighborhood *of a set of vertices $S \subseteq A$ is defined as $N(S) = \{v \in B : \exists u \in S. \{u, v\} \in E\}$.*

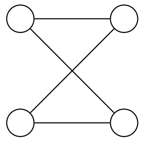

Figure 8: A bipartite graph.

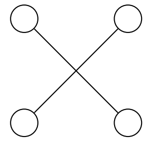

Figure 9: One perfect matching.

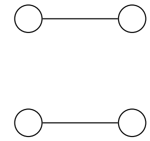

Figure 10: Another perfect matching.

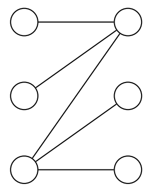

Figure 11: Bipartite graph with no perfect matching.

Theorem 4 (Hall's Marriage Theorem (1935))**.** *A bipartite graph $G = (A \cup B, E)$ has an A-perfect matching if and only if for any subset S of A $(S \subseteq A)$, the total number of vertices in the neighborhood of S is at least the number of vertices in S, i.e.,*

$$\forall S \subseteq A.\ |N(S)| \geq |S|.$$

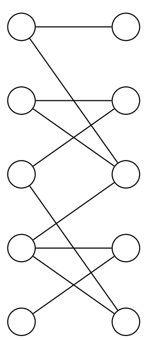

Figure 12: A bipartite graph.

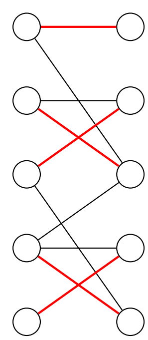

Figure 13: A perfect matching of the graph in Figure 12 highlighted in red.

5 Summary

In this lecture, we introduced notation and basic properties of graphs. We introduced special types of graphs, including the empty graph, complete graph, and cycle graph. Finally, we studied bipartite graphs and considered matchings in this context.