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Lecture 5

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1 Overview

In this lecture, we study predicate logic, which builds on propositional logic. We end with a brief discussion of Gödel's incompleteness theorems and their implications.

2 Predicate Logic

The general idea of predicate logic is similar to propositional logic—there are variables and operators—but now we have predicates. Recall that a predicate is a proposition whose truth value depends on certain parameters. For example,

$$E(x) = \text{“}x \text{ is even.”} \quad \text{and} \quad O(x) = \text{“}x \text{ is odd.”}$$

are two examples of predicates. We use quantifiers to interact with predicates. There are two quantifiers that we should know: the universal quantifier (denoted by $\forall$, pronounced "for all") and the existential quantifier ($\exists$, "there exists"). By using quantifiers with predicates (and other logical symbols), we can create predicate formulas such as the following:

$$P = (\forall x \in \mathbb{Z}^+ .\, E(x) \vee O(x)).$$

(Recall that $\mathbb{Z}^+$ denotes the set of positive integers.) In words, proposition P states that every positive integer is even or odd (or possibly both, although we know that's never the case).

2.1 Negating Quantifiers

Since P (defined above) is not only a predicate formula but also a proposition, P can be negated. To do this, we use a version of De Morgan's Laws for quantifiers. These laws are the following:

$$\neg(\forall x.\, Q(x)) = \exists x.\, \neg Q(x)$$
$$\neg(\exists x.\, Q(x)) = \forall x.\, \neg Q(x).$$

In other words, the $\neg$ is distributed and the quantifier "flips" (similarly to the regular De Morgan's Laws). Now why are these laws true, and why are they a version of De Morgan's Laws?

Imagine that the variable x can only take one of three possible values: a, b, or c. Then notice that

$$\forall x.\, Q(x) = (Q(a) \wedge Q(b) \wedge Q(c)),$$

which means

$$\begin{aligned} \neg(\forall x.\, Q(x)) &= \neg(Q(a) \wedge Q(b) \wedge Q(c)) \\ &= \neg Q(a) \vee \neg Q(b) \vee \neg Q(c). && \text{(De Morgan's for } \wedge\text{)} \end{aligned}$$

This last expression states that $Q(x)$ is false for at least one of the three possible values of x; in other words, this formula is equivalent to $\exists x, \neg Q(x)$. The other direction is similar:

$$\begin{aligned} \neg(\exists x.\, Q(x)) &= \neg(Q(a) \vee Q(b) \vee Q(c)) \\ &= \neg Q(a) \wedge \neg Q(b) \wedge \neg Q(c) && \text{(De Morgan's for } \vee) \\ &= \forall x.\, \neg Q(x). \end{aligned}$$

So the negation of our proposition P is the following:

$$\begin{aligned} \neg P &= \neg(\forall x \in \mathbb{Z}^+.\, E(x) \vee O(x)) \\ &= (\exists x \in \mathbb{Z}^+.\, \neg(E(x) \vee O(x))) && \text{(De Morgan's for quantifiers)} \\ &= (\exists x \in \mathbb{Z}^+.\, \neg E(x) \wedge \neg O(x)). && \text{(De Morgan's for } \vee) \end{aligned}$$

In English, $\neg P$ states that there is a positive integer that is neither even nor odd, which of course is false. This confirms the fact that the proposition P is always TRUE.

2.2 Combining Quantifiers

In propositional logic, we can combine multiple $\wedge$'s, $\vee$'s, and $\neg$'s to create complicated propositional formulas. Similarly, we can combine $\forall$'s and $\exists$'s to create complicated predicate formulas. Consider the following statement:

Every even number greater than 2 is the sum of two primes.

This proposition is known as *Goldbach's conjecture*, and despite its simplicity, it is one of the oldest and most well-known unsolved problems in mathematics. Let's reformulate the statement into predicate logic: let E denote the set of even numbers greater than 2, and let P denote the set of prime numbers. Also, define R as the following predicate:

$$R(a,b,c) \leftrightarrow (a = b + c).$$

Then Goldbach's conjecture states the following:

$$\forall x \in E\, \exists p \in P\, \exists q \in P.\, R(x,p,q).$$

Notice that this predicate formula has three quantifiers: one $\forall$ and two $\exists$'s. For notational convenience, we often collect variables under the same quantifier whenever possible, so this predicate formula is equivalent to the following:

$$\forall x \in E\, \exists p,q \in P.\, R(x,p,q).$$

2.3 Order of Quantifiers

In Goldbach's conjecture, and in most predicate formulas, the order of quantifiers matters. For example, consider the following alteration of Goldbach's conjecture:

$$\exists p,q \in P\, \forall x \in E.\, R(x,p,q).$$

In English, this proposition says that there are two prime numbers whose sum is equal to every even number greater than two. Of course, this is false because the sum of two numbers can only be a single number.

So swapping the order of quantifiers does not create an equivalent predicate formula, but that's not all. Consider the following predicate formulas:

$$A = (\forall x\, \exists y.\, R(x, y))$$
$$B = (\exists y\, \forall x.\, R(x, y)).$$

Goldbach's conjecture has the same form as A, and we just saw that the corresponding B is false. So even if Goldbach's conjecture is true, it is not the case that A implies B. However, it is the case that B implies A. (In other words, the proposition "$B \to A$" is true.)

To illustrate why this statement is true, let us again imagine that the variable x can only be one of three values, which we denote by x_1, x_2, and x_3. Then B implies there is some value of y, which we denote by y^*, such that

$$R(x_1, y^*) \wedge R(x_2, y^*) \wedge R(x_3, y^*).$$

To prove that A is true, we must show that for any possible value of x, there exists some value of y such that $R(x, y)$ is true. But B implies the existence of y^*, which we can exhibit as the value of y regardless of the value of x we are given. In other words, A is equivalent to

$$\exists y_1, y_2, y_3.\, R(x_1, y_1) \wedge R(x_2, y_2) \wedge R(x_3, y_3),$$

and we prove A by setting $y_1 = y_2 = y_3 = y^*$. On the other hand, if A is true, then the appropriate values y_1, y_2, y_3 exist but may not be the same. Thus, B is not necessarily true, because B demands a single value of y that works for all possible values of x. The takeaway is the following: the order of quantifiers in predicate formulas is important to consider when writing proofs.

3 Gödel's Incompleteness Theorems

In this section, we informally state Gödel's incompleteness theorems and provide some brief commentary. Although a formal proof is outside the scope of this class, these theorems relate to the foundations of mathematics, so we'd like to have a rough idea of what they mean.

Recall from Lecture 1 that ZFC is a set of axioms that form the basis of modern-day mathematics. But ZFC is only one system; there are other axiomatic systems that mathematicians can use as well. Any reasonable system, however, should satisfy the following properties:

- Soundness: The system should not be able to prove false statements.
- Completeness: The system should be able to prove every true statement.

Notice that satisfying exactly one of the properties is easy: for example, a system that does not allow us to prove anything would not be able to prove false statements.

In the early 20th century, mathematicians sought a system that is both sound and complete. However, in 1931, Kurt Gödel showed that this is impossible. More precisely (but still informally), Gödel proved the following:

If X is a sound logical system, then X is not complete.

To show that X is not complete, Gödel had to state a proposition that any sound system X would not be able to prove. That proposition is the following:

The system X is complete.

In other words, Gödel showed that any sound logical system cannot prove its own completeness, and therefore, cannot be complete. This statement is known as Gödel's *second* incompleteness theorem. Gödel *first* incompleteness theorem came to a similar conclusion but was conditioned on the logical system being related to the arithmetic of numbers. It (informally) states the following:

If a logical system X can perform arithmetic on the positive integers and is sound, then X is incomplete.

4 Summary

In this lecture, extended our study of logic to quantifiers and predicate formulas and saw a brief commentary on Gödel's incompleteness theorems.