1 Overview

In this lecture, we introduce sequences and Cartesian products. We also define relations and study their properties.

2 Sequences and Cartesian Products

Recall that two fundamental properties of sets are the following:

1. A set does not contain multiple copies of the same element.

2. The order of elements in a set does not matter.

For example, if $S = \{1, 4, 3\}$ then S is also equal to $\{4, 1, 3\}$ and $\{1, 1, 3, 4, 3\}$. If we remove these two properties from a set, the result is known as a sequence.

Definition 1. *A* sequence *is an ordered collection of elements where an element may repeat multiple times.*

An example of a sequence with three elements is $(1, 4, 3)$. This sequence, unlike the statement above for sets, is *not* equal to $(4, 1, 3)$, nor is it equal to $(1, 1, 3, 4, 3)$. Often the following shorthand notation is used to define a sequence:

$$s_i = \frac{1}{2^i} \ \forall i \in \mathbb{N}$$

Here the sequence $(s_1, s_2, \dots)$ is formed by plugging $i = 1, 2, \dots$ into the expression to find s_1, s_2, and so on. This sequence is $(1/2, 1/4, 1/8, \dots)$. We now define a new set operator; the result of this operator is a set whose elements are two-element sequences.

Definition 2. *The* Cartesian product *of sets A and B, denoted by $A \times B$, is a set defined as follows:*

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Notice that each element of $A \times B$ is a sequence containing two elements—the first from A, and the second from B. For example,

$$A = \{1, 5, 3\}, \quad B = \{0, 2\}.$$

Then

$$A \times B = \{(1, 0), (1, 2), (5, 0), (5, 2), (3, 0), (3, 2)\}.$$

Remember than in any set, we can freely alter the order of elements, but in a sequence, we cannot alter the order of elements. Thus, we can also write $A \times B$ as

$$A \times B = \{(5,0), (1,2), (3,2), (5,2), (3,0), (1,0)\},$$

but since $(0,1)$ is not an element of $A \times B$, we have

$$A \times B \neq \{(0,1), (1,2), (5,0), (5,2), (3,0), (3,2)\}.$$

Observe that $\mathbb{R}^2$ is shorthand for $\mathbb{R} \times \mathbb{R}$, and it represents the Cartesian plane that we often use to plot functions such as $y = 3x^2 + 1$.

Suppose A and B are finite sets. Recall that the cardinality of set A, denoted $|A|$, is the number of elements in A; similarly, $|B|$ is the cardinality of set B. What is the cardinality of their Cartesian product, i.e., $|A \times B|$? There are $|A|$ choices for the first term of each pair and $|B|$ choices for the second term of each pair. Thus, for finite sets A, B:

$$|A \times B| = |A| \cdot |B|.$$

We can also take the Cartesian product of a set and itself. A common notation used for expressing this is as follows:

$$A^n = \underbrace{A \times A \times ..... \times A}_{n \text{ times}}.$$

3 Relations

Definition 3. *A binary relation between two sets A and B is a set R of ordered pairs (a, b) consisting of elements $a \in A$ and $b \in B$. In other words, $R \subseteq A \times B$. If $(a, b) \in R$, we often write it as aRb.*

Example 1: Define relation R as a positive integer and twice its value. Thus, $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. We can explicitly write down elements of R:

$$R = \{(1,2), (2,4), (3,6), \dots \}$$

In set builder notation:
$$R = \{(a, 2a) : a \in \mathbb{Z}^+\}$$

Note that for every relation, we may not be able to describe it with a similar succinct formulation. Any subset of the Cartesian product is a valid binary relation.

Example 2: Suppose $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e, f\}$, and relation $R = \{(1,a), (2,c), (3,f), (3,c)\}$. We can draw this relation as a map from A to B, given in Figure 1.

The *out-degree* of an element $a \in A$ is the number of pairs of R in which a appears (the number of arrows leaving a in the map). Similarly, the *in-degree* of an element $b \in B$ is the number of pairs of R in which b appears (the number of incoming arrows to b in the map). In Example 2, the out degree of $3 \in A$ is 2 and the in degree of $e \in B$ is zero. Table 1 gives a characterization of a relation based on the in-degrees and/or out-degrees in its map.

For a relation to be a function, notice that for each $a \in A$, a cannot appear in more than one pair of the relation. In a total relation, any $a \in A$ must appear in at least one pair. If both of these
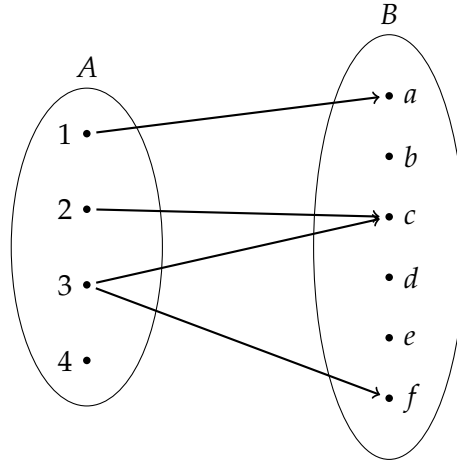
Figure 1: Figure representing relation in Example 2.

property of relation	type of relation
out-degree of all elements ≤ 1:	function
out-degree of all elements ≥ 1:	Total Relation
in-degree of all elements ≤ 1:	"INTO" (injective)
in-degree of all elements ≥ 1:	"ONTO" (surjective)

Table 1: Classes of relations

properties are satisfied the relation is a total function. Usually, we will assume functions are always total functions. When this is not the case, we will specify that the function is partial. If a relation is both injective and surjective, the relation is a *bijective* relation. If all 4 properties are satisfied, the relation is a bijective function.

Observation 1. *For finite sets A and B such that there exists a bijective function between A and B, it must be that $|A| = |B|$.*

This observation is intuitive for finite sets, but what about infinite sets?

Example 3: Let $A = \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $B = \{2, 4, 6, \dots\}$ be the set of even positive integers. It is clear that B is a strict subset of A, i.e., $B \subset A$. Define the following relation:

$$R = \{(a, 2a) : a \in A\}.$$

For each $a \in \mathbb{Z}^+$ it appears exactly once in relation R, so this is a total function. Additionally, each b appears once in R so this is a bijective function between A and B. This should give you pause; how did we come up with a bijection between two sets where one is a strict subset of the other? Crucially, these are infinite sets. We will see later in the course that comparing the sizes of infinite sets must be treated with care.

For any relation $R \subseteq A \times B$ we refer to A as the domain and B as the codomain. We also define:

$$\text{Range}(R) = \{b \in B : \exists (a, b) \in R\}.$$

The range of a relation is the subset of the codomain that participates in the relation. Similarly we define:

$$\text{Support}(R) = \{a \in A : \exists (a, b) \in R\}.$$

The support is the subset of the domain participating in relation R. Let's consider a few examples.

Example 4: Let us define a relation $R \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as the following:

$$aRb \text{ if } a \in \mathbb{Z}^+ \text{ and } b = -a.$$

In other words, $R = \{(1, -1)(2, -2), \dots\}$. Let's consider some properties of this relation:

domain:	$\mathbb{R}$		Total:	No
codomain:	$\mathbb{R}$		Function:	Yes
support:	$\mathbb{Z}^+$		Injective:	Yes
range:	$\mathbb{Z}^-$		Surjective:	No

While we define the relation in terms of the real numbers, only the positive integers appear from A. Thus, the support of R is the set of positive integers. Only the negative integers from $B = \mathbb{R}$ appear in the relation, this set is the range. Notice that the domain and support are not the same set, so there must be some elements of the domain not participating in the relation. Thus, this is not a total relation. An element either participates once in the relation, or not at all. This implies the relation is a function and that it is injective. The relation is not surjective. We can see this by observing that the range and codomain are different sets.

Example 5: Let's take a look at another relation. We define relation R such that $R \subseteq \mathbb{Q}^+ \times \mathbb{R}$ and

$$R = \left\{ \left(\frac{a}{b}, a + b \right) : \frac{a}{b} \in \mathbb{Q}^+ \text{ and } gcd(a, b) = 1 \right\}.$$

domain:	$\mathbb{Q}^+$		Total:	Yes
codomain:	$\mathbb{R}$		Function:	Yes
support:	$\mathbb{Q}^+$		Injective:	No
range:	$\mathbb{Z}^+ \setminus \{1\}$		Surjective:	No

Consider the range of R. Clearly the range is a subset of the positive integers, but notice that for any positive integer $k > 1$, $(\frac{k-1}{1}, k) \in R$. Does the range include 1? The rational number $\frac{0}{1}$ would produce $0 + 1 = 1$, but this is not a positive rational. So, our range is all positive integers except 1. Because the $gcd(a, b) = 1$, each rational number has out-degree exactly 1, so R is a total function. However, R is not surjective since the codomain and range are different sets. Additionally, R is not injective because the in-degree of b in the range could greater than 1. Consider the positive rationals $\frac{1}{4}$ and $\frac{2}{3}$. Both of these map to the integer 5, so the in-degree of 5 is greater than 1.

3.1 Composing Relations

Consider the following theorem.

Theorem 2. *For sets A, B, C if R_1 is a surjective relation from A to B and R_2 is a surjective relation from B to C, then there exists a surjective relation from A to C.*

Before we prove this, consider the map in Fig. 2. Intuitively, we can construct a surjective relation between A and C by following the paths from elements of A passing through elements of B to elements of C. In Fig. 2, the arrows from A to B represent R_1 and the arrows from B to C represent R_2. To find a surjective relation from A to C, we follow the two paths in red and say that the element of A at the beginning of the path is related to the element of C at the end of the path.
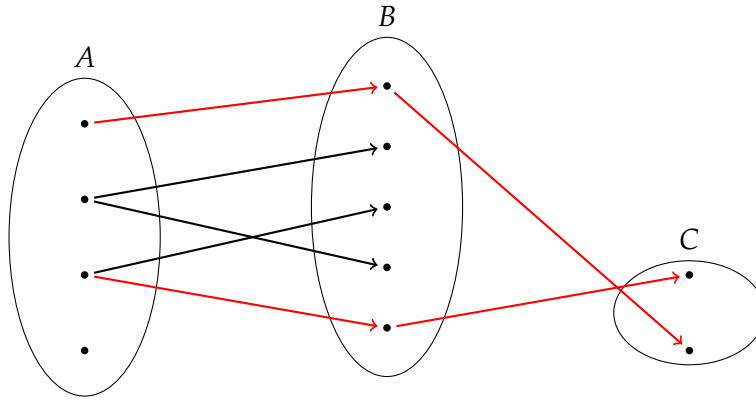

Figure 2: Composing relations.

Proof of Theorem 2. Given $R_1 \subseteq A \times B$ is surjective and $R_2 \subseteq B \times C$ is surjective, we will construct a new relation R_3 such that $R_3 \subseteq A \times C$ and

$$R_3 = \{(a, c) : \exists b \in B \text{ s.t. } aR_1b \text{ and } bR_2c\}.$$

Thus, the pair (a, c) is in our relation if there exists $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. We must prove R_3 is surjective. Let $c \in C$ be an arbitrary element. Since R_2 is surjective, there exists $b \in B$ such that $(b, c) \in R_2$. Since R_1 is surjective, there exists $a \in A$ such that $(a, b) \in R_1$. By our definition of R_3, $(a, c) \in R_3$. Because we started with an arbitrary element in C and showed it appears in at least one pair of R_3, we can conclude that R_3 is surjective. $\qquad\square$

The definition of R_3 in Theorem 2 is a composition of relations.

Definition 4. *For sets A, B, C if $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ then the* composition $R_2 \circ R_1$ *is defined as follows:*

$$R_2 \circ R_1 = \{(a, c) : \exists b \in B \text{ s.t. } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}.$$

Thus, $R_2 \circ R_1$ is a relation defined from A to C ($R_2 \circ R_1 \subseteq A \times C$). In Theorem 2 we showed that taking a composition of two surjective relations results in another surjective relation. We will prove a similar theorem about injective relations, functions, and total relations.

Theorem 3. *For sets A, B, C if $R_1 \subseteq A \times B$ is an injective relation and $R_2 \subseteq B \times C$ is an injective relation, then $R_2 \circ R_1$ is an injective relation.*

Proof. Consider an arbitrary $c \in C$. Since R_2 is an injective relation, either c does not participate in R_2, or c participates in the relation exactly once. If $(b, c) \notin R_2$ for any b, then c will not appear in relation $R_2 \circ R_1$ by definition. Otherwise, there exists a b such that $(b, c) \in R_2$. This is the only element of B which maps to c. We now consider this b.

Since R_1 is an injective relation, either b does not participate in R_1, or b participates in the relation exactly once. If $(a, b) \notin R_1$ for any a, then c will not appear in relation $R_2 \circ R_1$ by definition. In this case, since nothing in A maps to b, there is no path from an element of A to c.

Otherwise, there exists an $a \in A$ such that $(a, b) \in R_1$. This is the only element of A which maps to b. Thus, the pair $(a, c) \in R_2 \circ R_1$. In all cases, an element $c \in C$ appears in the relation exactly once or not at all. This implies that $R_2 \circ R_1$ is injective. $\square$

Theorem 4. *For sets A, B, C if $R_1 \subseteq A \times B$ is a function and $R_2 \subseteq B \times C$ is a function, then $R_2 \circ R_1$ is a function.*

Proof. Consider an arbitrary $a \in A$. Because R_1 is a function, a appears in a pair $(a, b) \in R_1$ at most one time. If a does not participate in R_1, then a will not map to any $c \in C$ in $R_2 \circ R_1$. Otherwise, a appears exactly once in a pair $(a, b) \in R_1$ for some $b \in B$. Consider this b.

Because R_2 is a function, b will appear in at most one pair (b, c). If b does not participate in R_2, then a will not map to any $c \in C$ in $R_2 \circ R_1$ since there is no path from a to c through this b. Otherwise, b appears exactly once in a pair $(b, c) \in R_2$ for some $c \in C$. In this case, $(a, c) \in R_2 \circ R_1$. This is the only time when a appears in $R_2 \circ R_1$. Thus, $R_2 \circ R_1$ is a function. $\square$

Theorem 5. *For sets A, B, C if $R_1 \subseteq A \times B$ is a total relation and $R_2 \subseteq B \times C$ is a total relation, then $R_2 \circ R_1$ is a total relation.*

Proof. Consider an arbitrary $a \in A$. Because R_1 is a total relation, a appears in at least one pair $(a, b) \in R_1$. It may appear in more than one pair of R_1, but we will consider one such b. Since R_2 is a total relation, b appears in at least one pair $(b, c) \in R_2$. Thus, $(a, c) \in R_2 \circ R_1$ by definition. Therefore, any $a \in A$ appears in at least one pair $(a, c) \in R_2 \circ R_1$ for some $c \in C$. This implies $R_2 \circ R_1$ is a total relation, as desired. $\square$

4   Summary

In this lecture, we reviewed the difference between sets and sequences. We also defined Cartesian products and binary relations. We defined properties of important classes of relations, and proved theorems using these properties.