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Lecture 8

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1 Overview

In this lecture, we continue studying relations and functions.

2 Relations and Functions

Recall that a relation is defined between two sets as a subset of the Cartesian product: $R \subseteq A \times B$ for sets A, B. We use the following shorthand:

- A bij B ("A has a bijection with B") if there exists a bijective total function $R \subseteq A \times B$.
- A surj B ("A has a surjection with B") if there exists a surjective total function $R \subseteq A \times B$.
- A inj B ("A has an injection with B") if there exists an injective total function $R \subseteq A \times B$.

Suppose A and B are finite sets. Then we have the following *cardinality rules*:

1. A bij $B \Leftrightarrow |A| = |B|$
2. A surj $B \Leftrightarrow |A| \geq |B|$
3. A inj $B \Leftrightarrow |A| \leq |B|$

Proof of Rule 1. Suppose A bij B. This means there exists $R \subseteq A \times B$ such that R is a bijective total function. Since R is a total function, every element of A maps to exactly one element of B. Thus, the total sum of in-degrees of elements of B is $|A|$. Since R is a bijection, every element $b \in B$ is mapped to exactly once. In other words, each $b \in B$ has in-degree one. Equivalently, for all $b \in B$, *in-degree(b)* $= 1$. We have the following:

$$|A| = \sum_{b \in B} \textit{in-degree(b)} = \sum_{b \in B} 1 = |B|$$

Thus, $|A| = |B|$ as desired. □

Proof of Rule 2. Suppose A surj B. This means there exists $R \subseteq A \times B$ such that R is a surjective total function. Since R is a total function, every element of A maps to exactly one element of B. Thus, the total sum of in-degrees of elements of B is $|A|$. Since R is a surjection, every element in B is either mapped to once or more than once. Equivalently, for all $b \in B$, *in-degree(b)* ≥ 1. We have the following:

$$|A| = \sum_{b \in B} \textit{in-degree(b)} \geq \sum_{b \in B} 1 = |B|$$

Thus, $|A| \geq |B|$ as desired. □

Proof of Rule 3. Suppose A inj B. This means there exists $R \subseteq A \times B$ such that R is a injective total function. Since R is a total function, every element of A maps to exactly one element of B. Thus, the total sum of in-degrees of elements of B is $|A|$. Since R is a injection, every element in B is either mapped once or not at all. Equivalently, for all $b \in B$, *in-degree(b)* ≤ 1. We have the following:

$$|A| = \sum_{b \in B} \textit{in-degree(b)} \leq \sum_{b \in B} 1 = |B|$$

Thus, $|A| \leq |B|$ as desired. □

Definition 1. *The* inverse relation*, denoted R^{-1}, of a relation R is the set of ordered pairs obtained by reversing those of R. If $R \subseteq A \times B$:*

$$aR^{-1}b \Leftrightarrow bRa.$$

Thus, $R^{-1} \subseteq B \times A$. Informally, R^{-1} is the relation obtained by changing the direction of arrows in the mapping diagram of R.

If R is a bijective total function, then R^{-1} is also a bijective total function. All out-degrees and in-degrees of R are exactly 1, so in R^{-1} the out-degrees become in-degrees, and vice versa. This leads to the following theorem:

Theorem 1. *A bij B if and only if B bij A.*

Proof. Suppose A bij B. By the cardinality rules, A bij B if and only if $|A| = |B|$. Equivalently, $|B| = |A|$. By the cardinality rules, $|B| = |A|$ if and only if B bij A. In addition to existence, if we have a bijection from A to B, we can find a bijection from B to A. Let R be a bijective total function from A to B. Taking the inverse of R results in a bijective total function from B to A, as previously observed. □

We present some observations when A, B are finite sets.

Observation 2. *If A surj B and B surj A, then A bij B.*

This follows from the cardinality rules: A surj B implies $|A| \geq |B|$ and B surj A implies $|B| \geq |A|$. Together, these imply $|A| = |B|$. From our cardinality rules we know $|A| = |B|$ if and only if A bij B.

Observation 3. *Either A surj B or B surj A.*

This follows from the cardinality rules. For any finite sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$. In the first case, $|A| \leq |B|$ implies B surj A and in the latter case $|B| \leq |A|$ implies A surj B.

Observation 4. *A surj B if and only if B inj A.*

Again, we use the cardinality rules. A surj B if and only if $|A| \geq |B|$. We know $|B| \leq |A|$ if and only if B inj A.

3 Properties of Relations on a Set

We say R is defined on set A if $R \subseteq A \times A$.

Example 1: Consider the set of positive integers, $\mathbb{Z}^+$. We can use comparators (such as $<$, $\leq$, $=$, $\geq$, $>$) to define relations on this set. Consider defining a relation using $\leq$. This means $(a, b) \in R \Leftrightarrow a \leq b$. For example, the pair $(3, 5) \in R$ since $3 \leq 5$, but $(5, 3) \notin R$ because $5 \nleq 3$.

Example 2: Consider relation R defined by $<$ on $\mathbb{Z}^+$.

Total?	Yes, $\forall x \in \mathbb{Z}^+ \; x < x + 1 \Rightarrow (x, x + 1) \in R$.
Function?	No, $3 < 4, 3 < 5, 3 < 6$, etc. An out-degree of an element could be larger than 1.
Injective?	No, $1 < 5, 2 < 5, 3 < 5, 4 < 5$. The in-degree of an element could be larger than 1.
Surjective?	No, there is no positive integer less than 1 so $(x, 1) \notin R$.

We introduce new properties of relations defined when the domain and codomain are the same set.

Definition 2. *A relation R on set A is* reflexive *when every element is related to itself. Formally,*

$$aRa \;\; \forall a \in A.$$

Definition 3. *A relation R on set A is* symmetric *when a relates to b if and only if b relates to a. In other words,*

$$aRb \;\Leftrightarrow\; bRa \;\; \forall a, b \in A.$$

Definition 4. *A relation R on set A is* transitive *if for every pair (a, b) if b also relates to some element c, then a must also relate to c. Formally,*

$$aRb \wedge bRc \Rightarrow aRc \;\; \forall a, b, c \in A.$$

Lemma 5. *Consider a relation R defined on set A. Suppose $\forall a \in A$, $\exists b \in A$ s.t. aRb. If R is both symmetric and transitive, then R is reflexive.*

Proof. Consider $a, b \in A$ such that $(a, b) \in R$. If $b = a$, we are done. Otherwise, by symmetry $(b, a) \in R$. By transitivity, since aRb and bRa, it must be that aRa. Since this holds for all $a \in A$, the relation is reflexive. □

Definition 5. *A relation R is* irreflexive *when no element a relates to itself. Formally,*

$$(a, a) \notin R \;\; \forall a \in A.$$

Note that a relation may be neither irreflexive nor reflexive.

Example 3: Suppose $A = \{1, 2\}$. Then $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Consider the following relations:

$R_1 = \{(1, 2), (2, 1)\}$	irreflexive since $(1, 1), (2, 2) \notin R_1$
$R_2 = \emptyset$	irreflexive since $(1, 1), (2, 2) \notin R_2$
$R_3 = \{(1, 1), (1, 2), (2, 2), (2, 1)\}$	reflexive since $(1, 1), (2, 2) \in R_3$
$R_4 = \{(1, 1)\}$	neither reflexive nor irreflexive

R_4 is not reflexive because $(2, 2) \notin R_4$. It is also not irreflexive because $(1, 1) \in R_4$.

Definition 6. *A relation R is* asymmetric *iff*

$$(a,b) \in R \Rightarrow (b,a) \notin R \;\; \forall a,b \in A.$$

Notice that if a relation is asymmetric it cannot contain pairs of the form (a,a) for any $a \in A$. Thus, if a relation is asymmetric then it is also irreflexive. Also notice that the only relation that is both symmetric and asymmetric is the empty set.

Example 4: Suppose $\mathbb{F}_2 = \{0,1\}$. Then $\mathbb{F}_2 \times \mathbb{F}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$. Consider the following relations:

$R_1 = \{(0,0)\}$	symmetric, transitive
$R_2 = \emptyset$	symmetric, asymmetric, transitive
$R_3 = \{((0,0),(0,1),(1,0),(1,1)\}$	symmetric, transitive
$R_4 = \{(1,1),(0,0)(1,0)\}$	transitive

In R_2, there are no pairs (a,b) in the relation for which we need to check the implications for symmetry, asymmetry, or transitivity. Recall that for a proposition $P \Rightarrow Q$, if P is false, then the implication is true. Notice that R_2 is not reflexive, since $(0,0),(1,1) \notin R_2$. Relation R_3 is the entire Cartesian product. Thus, it is symmetric and transitive since every possible pair is present. Relation R_4 is not symmetric, as $(1,0) \in R_4$, but $(0,1) \notin R_4$. It is also not asymmetric, as it has pairs $(0,0)$ and $(1,1)$. To establish that R_4 is transitive, consider $(1,1)$ and $(1,0)$. Transitivity implies that $(1,0) \in R_4$, which is true. Also consider $(1,0)$ and $(0,0)$; we have that $(1,0) \in R_4$. Thus, for any pairs of the form $(a,b),(b,c) \in R_4$ we know that $(a,c) \in R_4$ as well.

Asymmetry is a strong condition; it does not allow any pairs of the form (a,a) in the relation. A slightly weaker, but similar condition is anti-symmetry.

Definition 7. *A relation R is* anti-symmetric *iff*

$$(a,b) \in R \textit{ and } a \neq b \Rightarrow (b,a) \notin R \; \forall a,b \in A.$$

Any relation which is asymmetric is also anti-symmetric. For example, a relation defined by $<$ on the positive integers is both asymmetric and anti-symmetric. However, $\leq$ is not asymmetric, but it is anti-symmetric. We see this by observing that for any positive integer x, we have $x \leq x$ but $x \not< x$.

4 Summary

In this lecture, we used cardinality rules to prove theorems about relations defined on finite sets. We also considered the inverse relation. Finally, we studied relations defined on a set, and learned the rules for reflexivity, irreflexivity, symmetry, asymmetry, anti-symmetry, and transitivity.