Due on April 6, 2019
40 points total

General directions:

All answers to non-programming questions must be typed, preferably using LaTeX. If you are unfamiliar with LaTeX, you are strongly encouraged to learn it. However, answers typed in other text processing software and properly converted to a PDF file will also be accepted. To submit your file, upload your PDF on Gradescope for assignment "HW X (PDF)." Handwritten answers or PDF files that cannot be opened will not be graded and will not receive any credit.

Finally, please read the detailed collaboration policy given on the course website. You are **not** allowed to discuss homework problems in groups of more than 3 students. **Failure to adhere to these guidelines will be promptly reported to the relevant authority without exception.**

Point values: Every problem has a specified amount of points which are awarded for the correctness of your solutions. In addition, each proof-oriented problem has an additional **style point**. In the homework handout, this is signified by a "+1" in the point value. To earn this point, your solutions should be clear, well organized, and easy to follow. This is to encourage not only perfectly correct solutions, but well presented ones.

Problem 1 (23+2 points)

(a) **(14+1 points)** Suppose $G = (A \cup B, E)$ is a bipartite graph. Recall the *neighborhood* of a set of vertices $S \subseteq A$ is defined as $N(S) = \{v \in B : \exists u \in S. \{u, v\} \in E\}$. The *deficiency* of $S \subseteq A$ is defined as $\Delta(S) = \max\{0, |S| - |N(S)|\}$. Then, the *deficiency* of G is the maximum deficiency of any subset of A:

$$\Delta(G) = \max_{S \subseteq A} \Delta(S).$$

Finally, recall a *matching* in G is a set of edges $M \subseteq E$ such that no two edges in M are incident on the same vertex.

Prove that the maximum number of edges in a matching contained in E is $|A| - \Delta(G)$.
(**Hint:** *Try forming a larger graph by adding $\Delta(G)$ new vertices to B and connect each new vertex to all vertices in A.*)

(b) **(9+1 points)** The standard deck of playing cards has 4 suits of 13 ranks, for a total of 52 cards. Suppose we remove an arbitrary 13 cards and place the remaining 39 cards in a grid with 3 rows of 13 cards per row. Prove that there is always a way to pick one card from each column so that at least 10 distinct ranks are selected.
(**Hint:** *Try to apply the result you proved in part* (a).)

Problem 2 (14+1 points)
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs. We say that G_2 contains a *copy* of G_1 if there exists a total injective function $f : V_1 \to V_2$ that satisfies the following property:

$$\forall u, v \in V_1. \{u, v\} \in E_1 \text{ implies } \{f(u), f(v)\} \in E_2.$$

In other words, the function f should map adjacent vertices in G_1 to adjacent vertices in G_2.

Let T be an arbitrary tree with k vertices. Prove the following: if G is a graph whose minimum degree is at least $k - 1$, then G contains a copy of T.

Hint: Recall that any tree has at least one leaf, i.e., a vertex with degree 1. Removing this vertex (and its single incident edge) yields a tree with fewer vertices. Using this fact, proceed with induction on k.