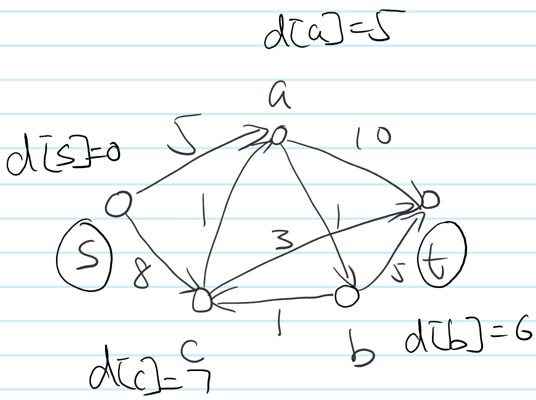
Lecture 12 Shortest Path

Wednesday, February 26, 2020 2:28 PM



$s \xrightarrow{5} a \xrightarrow{10} t$ 15

$s \xrightarrow{5} a \xrightarrow{1} b \xrightarrow{5} t$ 11

$s \xrightarrow{5} a \xrightarrow{1} b \xrightarrow{1} c \xrightarrow{3} t$ 10

s – t shortest path: go from s to t

single-source shortest path: shortest paths from s to every other vertex

all pairs shortest path: shortest paths between every pair.

- dynamic programming

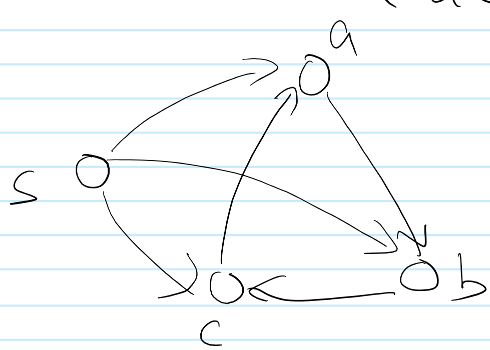
  state: let d[v] be the length of shortest path from s to v.

$$d[v] = \min_{(u,v)\in E} w(u,v) + d[u]$$

  length of last edge; distance to u.

  for example graph

$$\underline{d[t]} = \min \begin{cases} d[a]+10 & 15 \\ d[b]+5 & 11 \\ d[c]+3 & \underline{\underline{10}} \end{cases}$$



- Dijkstra's algorithm

  - maintain a set of visited vertex V
  (also the vertices that we have computed shortest path for)

  (initially {s})

(initially $\{s\}$)

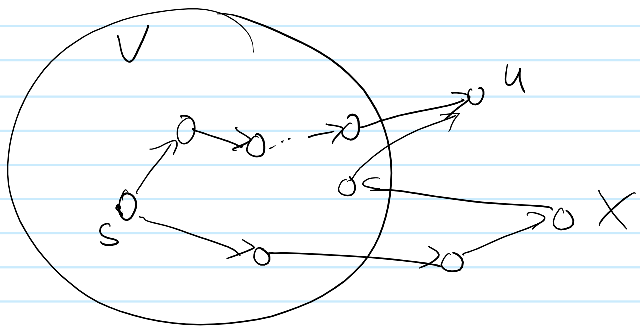
- maintain dis array

  for vertices that are visited ($u \in V$)

  $$dis[u] = d[u] = \text{length of shortest path from } s \text{ to } u.$$

  for vertices not visited ($u \notin V$)

  $dis[u]$ = length of shortest path from s to u, only use vertices in V as intermediate vertices

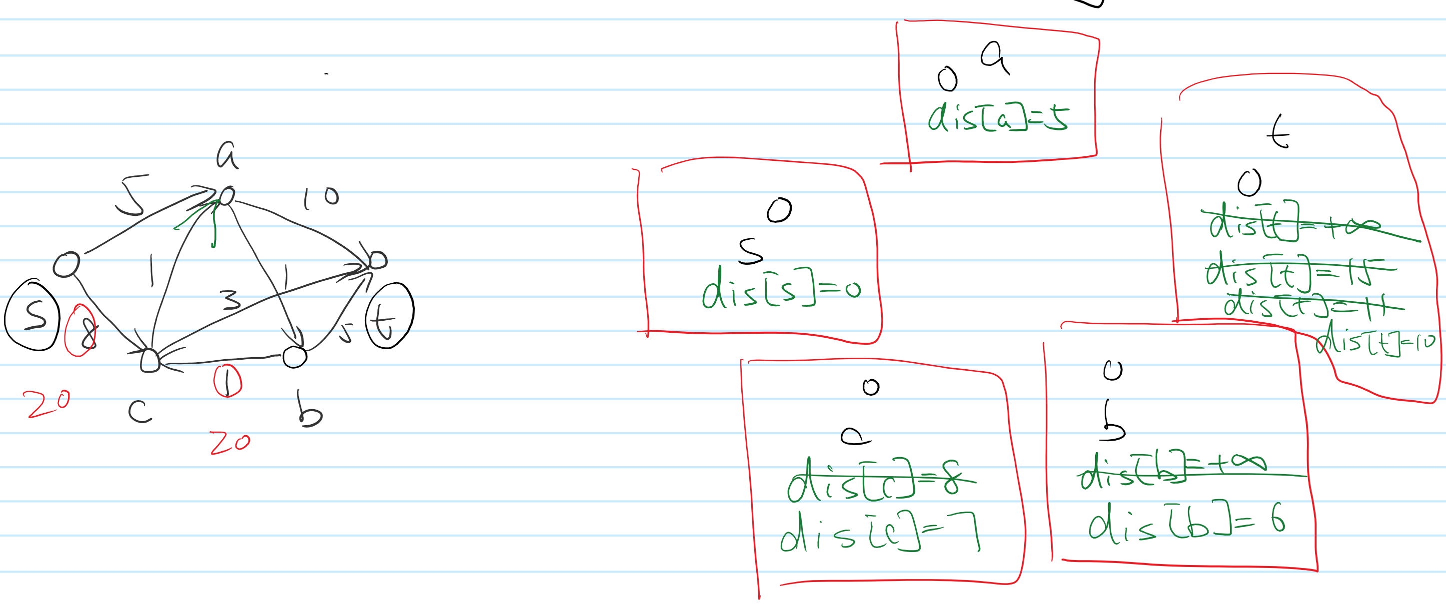


- at every iteration

  select $u \notin V$ such that $dis[u]$ is smallest.

  Claim: for this u $dis[u] = d[u]$

  add u to V, update dis array.



- Proof of correctness:

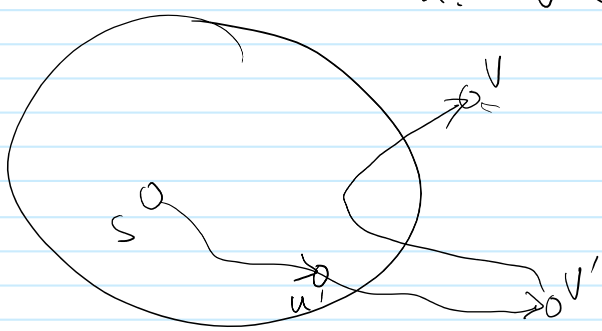
Main step: prove the claim that for vertex v with smallest $dis[v]$ among $\notin V$

$$d[v] = dis[v]$$

assume towards contradiction that there is a path from s to v with length shorter than $dis[v]$

by IH, this shorter path must use vertices that are not visited as intermediate vertices.

let v' be the first vertex on the path s.t. $v' \notin V$



by IH, we know distance from s to v' is at least $dis[v']$

but $dis[v] \leq dis[v']$ by choice of algorithm, so length of this path cannot be smaller than $dis[v]$.

need to prove: in the next iteration, $dis[v]$ is correctly maintained.