Lecture 12: Shortest Path

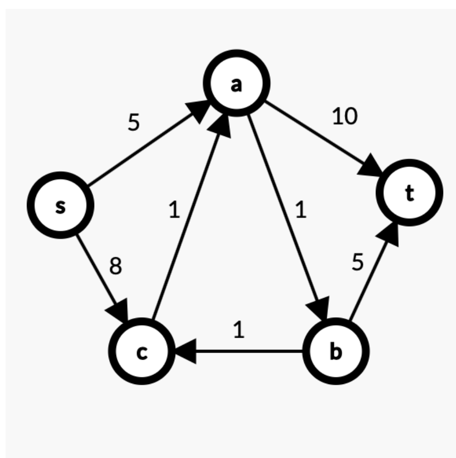
Scriber: Xiaoming Liu

Feb 9, 2020

1 Shortest Path

1.1 s-t Shortest Path

Using the following graph as an example. Given that s is the starting node and t is the target node. Find the s-t shortest path.



The possible paths from s to t are:

1. s $\xrightarrow{5} a \xrightarrow{10} t$.The total cost is 15.
2. s $\xrightarrow{5} a \xrightarrow{1} b \xrightarrow{5} t$.The total cost is 11.
3. s $\xrightarrow{5} a \xrightarrow{1} b \xrightarrow{1} c \xrightarrow{3} t$. The total cost is 10.

1.2 Single Source Shortest Path

Problem Statement: Find the shortest path from a single source s to every other vertex in the graph.

State: Let d[v] be the length of shortest path from s to v.
Transition Function

$$d[v] = min_{(u,v)\in E}(w(u,v) + d[u])$$

In which $w(u,v)$ is the length of last edge and $d[u]$ is the distance from s to u.

Take the graph above as an example.

$$d[t] = min \begin{cases} d[a] + 10, & 15 \\ d[b] + 15, & 11 \\ d[c] + 3, & 10 \end{cases}$$

1.3 Dijkstra's Algorithm

Maintain a set of visited vertices(the vertices that we have computed shortest path for) V. The set is initialized as s, which contains only the source node.
We also need to maintain a distance array.

1. For vertices that are visited. $(u \in V)$.

   dis[u] = d[u] = Length of shortest path from s to u.

2. For vertices not visited $(u \notin V)$

   dis[u] = d[u] = Length of the shortest path from s to u, only use vertices in V as intermediate vertices.

At every iteration, select $u \notin V$ such that dis[u] is smallest. Add u to V, update the dis array.
Proof of Correctness
The main step here is to prove the claim that for vertex v with smallest dis[v] among the vertices not in set V, d[v] = dis[v].
Assume towards contradiction that there is a path from s to v with length shorter than dis[v]. By the inductive hypothesis, the shorter path must use vertices that are not visited as intermediate vertices. Let v' be the first vertex on the path such that $v' \notin V$. By induction hypothesis, wee know distance from s to v' is at least dis[v'], but $dis[v]dis[v']$ by choice of the algorithm, so length of this path cannot be smaller than $dis[v]$.