Lecture 13 Bellman-Ford, Minimum Spanning Tree

Thursday, March 5, 2020 9:47 AM

- shortest path with negative edge length
  - arbitrage example

assume $(u_1, u_2, \cdots, u_k)$ is a negative cycle.

$C(u_i, u_{i+1})$: 1 unit of $u_{i+1} = C(u_i, u_{i+1})$ units of u_i

$u_1 \to u_2 \longrightarrow \cdots \longrightarrow u_k$

if we start with 1 unit of u_1, we get

$$\frac{1}{C(u_1,u_2)\,C(u_2,u_3)\cdots C(u_{k-1},u_k)} \text{ units of } u_k$$

$u_k \to u_1$ $C(u_k, u_1)$, get $\boxed{\frac{1}{C(u_1,u_2)\,C(u_2,u_3)\cdots C(u_{k-1},u_k)\,C(u_k,u_1)}}$ units of u_1

1

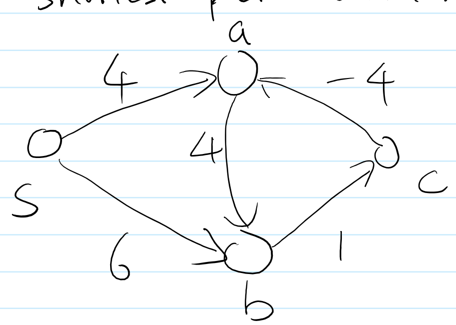
$$w(u_i, u_{i+1}) = \log(C(u_i, u_{i+1}))$$

$$\underbrace{\sum_{i=1}^{k-1} w(u_i, u_{i+1}) + w(u_k, u_1)}_{\text{total length of the cycle}} < 0$$

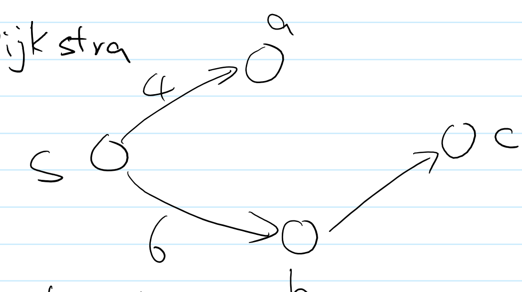
$$\sum_{i=1}^{k-1} \log C(u_i, u_{i+1}) + \log C(u_k, u_1) < 0$$

$$C(u_1,u_2)\,C(u_2,u_3)\cdots C(u_{k-1},u_k)\,C(u_k,u_1) < 1$$

- shortest path with negative edge but no negative cycle.

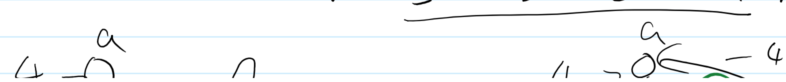


Dijkstra



thinks shortest path from s to a has length 4.

However $\underline{s \to b \to c \longrightarrow a}$ has length $6+1-4=3$



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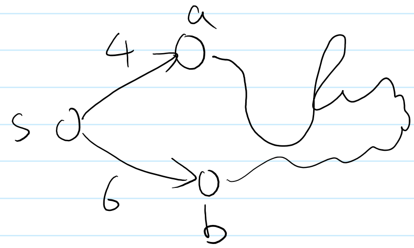
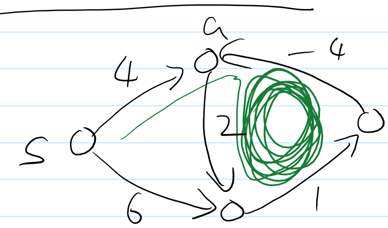
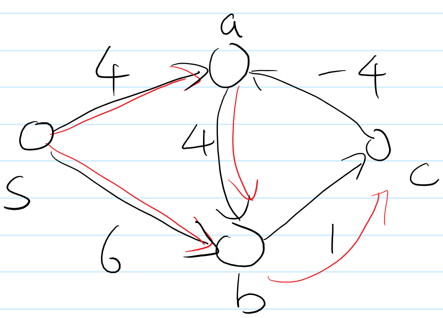




- Bellman-Ford



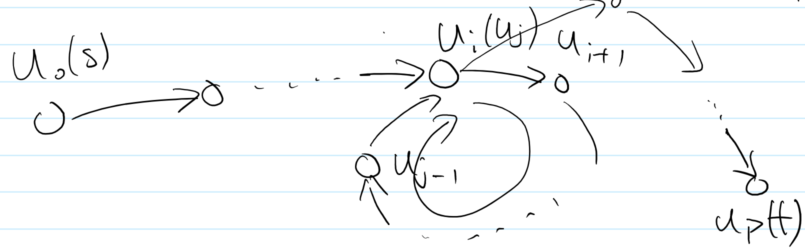
$d(u,i)$	s	a	b	c
0	0	$+\infty$	$+\infty$	$+\infty$
1	0	4	6	$+\infty$
2	0	4	6	7
3	0	3	6	7
4	0	3	6	7

- If a graph has no negative cycle, then for any pair of vertices (s,t) (such that it's possible to go from s to t), there exists a shortest path from s to t using at most $\underline{n-1}$ edges.

- Proof: assume towards contradiction that there is a shorter path.

$$(\underset{s}{u_0}, u_1, \dots, \underset{t}{u_p}) \qquad p \geq n$$

there is at least 1 vertex that appeared more than once (u_i, u_j)



consider the path

$(u_0, \dots, u_i, u_{j+1}, \dots, u_p)$

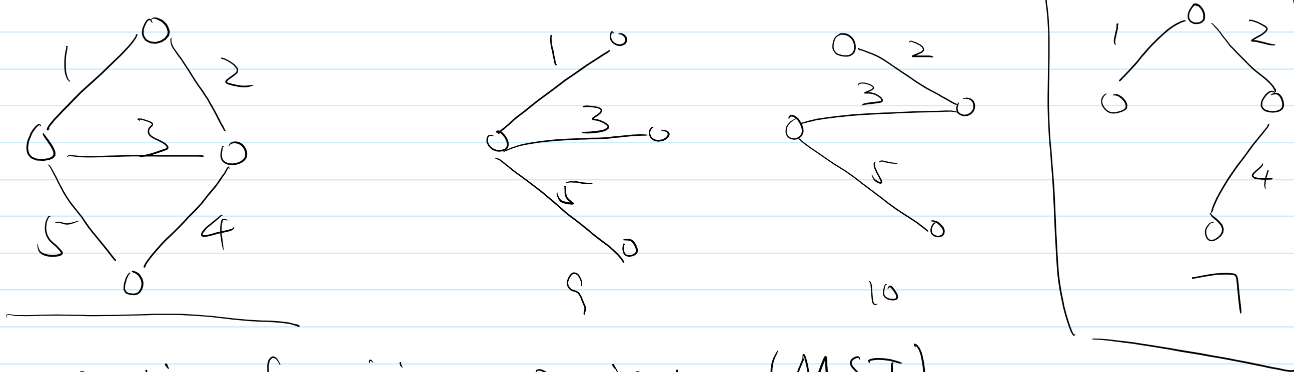
this path uses fewer edges but has same or shorter length.

repeating this argument gives a path with at most $n-1$ edges, and has same or shorter length compared to $\underline{(u_0, u_1, \dots, u_p)}$, this is a contradiction $\square$

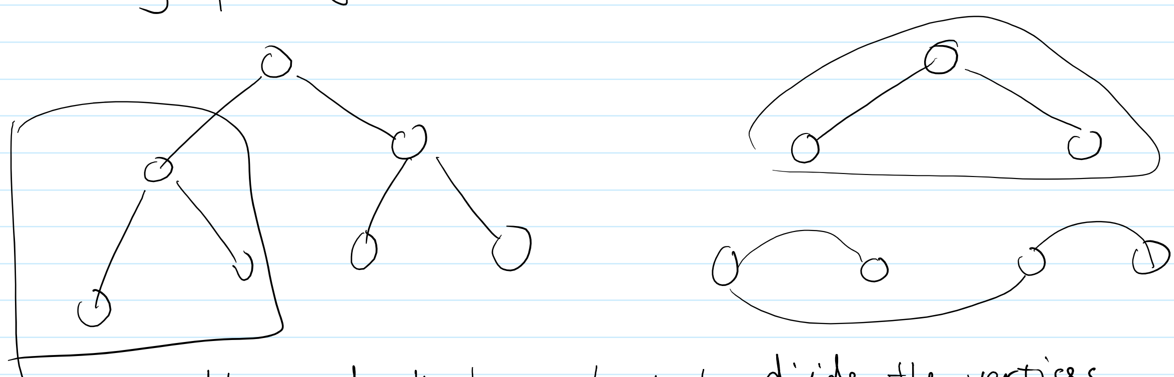
to $(u_0, u_1, \ldots, u_p)$, this is a contradiction $\square$

- minimum spanning tree
  - spanning tree

    a spanning tree T of a graph $G=(V,E)$ is a subset of edges of size $n-1$, such that all pairs of vertices are connected by these edges.

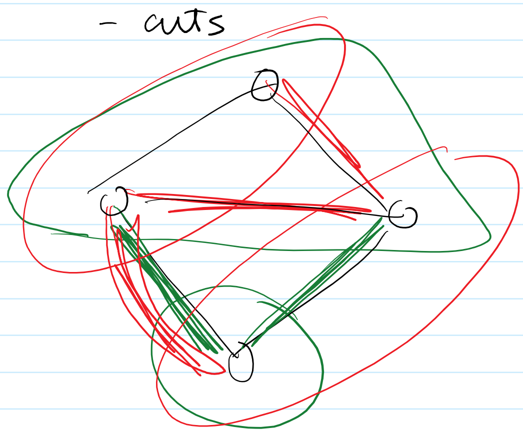


- properties of minimum spanning tree (MST)
  - any part of MST is also an MST.



problem: don't know how to divide the vertices

- cuts



construct a cut: choose a set of vertices S, remaining vertices $\overline{S}$

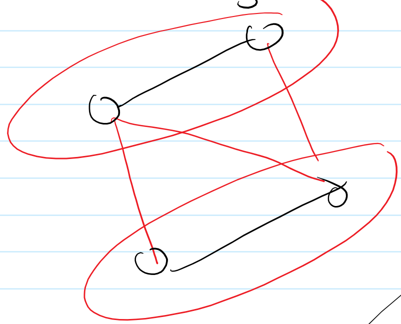
$$c(S,\overline{S}) = \{\underline{(u,v)} \in E \mid u \in S, v \in \overline{S}\}$$

if $(u,v) \in E$, $u \in S$, $v \in \overline{S}$, say edge (u,v) "crosses" the cut $(S,\overline{S})$

- Cut and connectivity

MST: connect all pairs of vertices

edges in a cut $(S, \bar{S})$: necessary to connect S with $\bar{S}$



Claim: For any spanning tree T, any cut $(S, \bar{S})$

there must be at least one edge (u, v) in

both T and $(S, \bar{S})$