Lecture 15: Linear Programming

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1 Linear Program

Most optimization problems are defined by three major components:

1. Set of Variables (Parameters)
2. Constraints
3. Objective

In linear programming, the components are defined as:

1. Variables: n real numbers, $x_1, x_2, ..., x_n \in \mathscr{R}$.
2. Constraints: a set of linear inequalities. For instance, these are a set of valid linear inequalities:
   (a) $2 * x_1 \geq x_2 - x_3$
   (b) $x_1 \leq x_5 + 10$
   (c) $c * x_1 \leq x_2$, given c is a known constant

   These are not valid linear inequalities:
   (a) $x_1 * x_2 \leq 1$
   (b) $log(x_1) + log(x_3) \leq 3$
   (c) $log(x_1) \leq 10$
3. Objective: Linear function over the variables.

An example of the linear program is illustrated as follows:

1. Variables: x, y
2. Constraints:

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 1$$

3. Objective: $max(2 * x + y)$

2 Solutions to linear program

Solution: An assignment of the variables
Feasible Solution: A solution that satisfies all constraints. For instance, for the example linear program problem above, some possible feasible solutions can be $(x,y) = (0,0),(1,0)),(\frac{1}{2},\frac{1}{2}),(\frac{1}{4},\frac{1}{2}),...$, while for instace, $(x,y) = (-1,-1)$ is not a feasible solution.
Optimal Solution: A feasible solution that achieves the best objective value. For instance, an optimal solution for the linear program above is $(1,0)$.

3 Geometric Interpretation

1. Variables: $(x_1,x_2,...,x_n)$ correspond to a point in n-dimensional space.

2. Constraints: Each linear inequality corresponds to a planes, and the group of linear inequalities indicates the intersection of half planes, which defines a feasible region.

3. Objective: The original objective can be rewritten as

$$max(\vec{c}\cdot x) = max\textstyle\sum_{i=1}^{n}(c_i x_i)$$

The point $\vec{c}$ represents a direction of gravity. When $\vec{c}$ points down, the lowest point of the feasible region gives the optimal solution.

4 Canonical Form

A linear program is in canonical form if it is of the form:

$$\begin{aligned} min < c,x > \\ s.t. x \geq b \\ x \geq 0 \end{aligned}$$

The form is equivalent to:

$$\begin{aligned} min\sum_{i=1}^{n}(c_i x_i) \\ \text{for every } j = 1,2,...,m \\ \sum_{i=1} nA_{j,i}x_i \geq b_j \\ \text{for every } i = 1,2,...,n \\ x_i \geq 0 \end{aligned} \tag{1}$$

The max and min in the optimization objective can be interchanged by taking the negative of the original objective. For instance, the following linear programming problems are equivalent:

1. $max(2x+y), x+y \leq 1$
2. $min(-2x-y), -x-y \geq 1$

5 Applying Linear Programming

5.1 Fractional Knapsack

Problem statement: Given a set of n items, in which item i has a weight of w_i and a value of v_i. Given a knapsack of fixed capacity c and you can put fractions of items into the knapsack. How to pack the knapsack so that it has the max value.
Variables: let x_i be the fraction of item i in the knapsack
Constraints:

1. Capacity constraint: $\sum_{i=1}^{n}(w_i x_i) \leq c$
2. Fraction constraint: $0 \leq x_i \leq 1, \forall i \in 1,2,..,n$

Objective: $max \sum_{i=1}^{n}(v_i x_i)$