Lecture 16 Linear Programming Duality

Wednesday, March 25, 2020 2:58 PM

- two player zero-sum games
  - each player has a set of strategies

    T_{row}, T_{col}
  - payoff matrix A, $A[i,j]$ $i \in T_{row}$, $j \in T_{col}$

    $A[i,j]$ = payoff to row player if row plays i
    col plays j

  - for mixed strategies

    payoff $P = \sum_{i \in T_{row}} \sum_{j \in T_{col}} A[i,j] \cdot P_{row,i} P_{col,j}$

    $\uparrow$ prob. row player plays strategy i

- LP duality
  - Example: $\underline{\min}$ $2x_1 - 3x_2 + x_3$

    $x_1 - x_2 \geq 1$ (1)

    $x_2 - 2x_3 \geq 2$ (2) optimal value OPT

    $-x_1 - x_2 - x_3 \geq -7$ (3)

    $x_1, x_2, x_3 \geq 0$

    OPT feasible

  - Prove $OPT \leq -1$

    answer: show a feasible solution with value -1

    $\boxed{(x_1, x_2, x_3) = (4, 3, 0)}$

    $\boxed{\begin{array}{c}(x_1, x_2, x_3) = (3, 2, 0) \\ 2x_1 - 3x_2 + x_3 = 0\end{array}}$

  - Prove $OPT \geq -1$

    $\underline{2.5} \times (1) + \underline{0.5} \times (3)$

    $2.5 \times (1)$ $\quad 2.5x_1 - 2.5x_2 \geq 2.5$

    $0.5 \times (3)$ $\quad -0.5x_1 - 0.5x_2 - 0.5x_3 \geq -3.5$

    $2x_1 - 3x_2 - 0.5x_3 \geq -1$

    $2x_1 - 3x_2 + x_3 \underset{x_3 \geq 0}{\geq} 2x_1 - 3x_2 - 0.5x_3 \underset{2.5 \times (1) + 0.5 \times (3)}{\geq} -1$

- generalizing the proof

  $\underline{\min}$ $2x_1 - 3x_2 + x_3$

  $x_1 - x_2 \geq 1$ (1) $\quad y_1$

$\min \; 2x_1 - 3x_2 + x_3$

$x_1 - x_2 \geq 1 \quad (1) \quad y_1$

$x_2 - 2x_3 \geq 2 \quad (2) \quad y_2$

$-x_1 - x_2 - x_3 \geq -7 \quad (3) \quad y_3$

$x_1, x_2, x_3 \geq 0$

$y_1 \times (1) + y_2 \times (2) + y_3 \times (3) \quad (*)$

① want to make sure $(*)$ is valid for all feasible x.

$\boxed{y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0}$

objective

$2x_1 - 3x_2 + x_3 \geq$ LHS of $(*)$ $\geq$ RHS of $(*)$

$$y_1(x_1 - x_2) + y_2(x_2 - 2x_3) + y_3(-x_1 - x_2 - x_3) \geq y_1 + 2y_2 - 7y_3$$

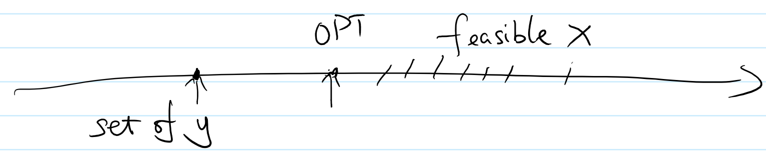
$$\underbrace{(y_1 - y_3)x_1 + (-y_1 + y_2 - y_3)x_2 + (-2y_2 - y_3)x_3}_{\text{LHS of } (*)} \geq y_1 + 2y_2 - 7y_3$$

② make sure: for every variable x_1, x_2, x_3

coefficient in objective $\geq$ coefficient in LHS of $(*)$

$$\boxed{\begin{aligned} y_1 - y_3 &\leq 2 \\ -y_1 + y_2 - y_3 &\leq -3 \\ -2y_2 - y_3 &\leq 1 \end{aligned}}$$

① + ② objective $\geq$ LHS of $(*)$ $\geq$ RHS of $(*)$



$\max$ RHS of $(*) = y_1 + 2y_2 - 7y_3$

primal

$\min \; 2x_1 - 3x_2 + x_3$

$x_1 - x_2 \geq 1 \quad (1)$

$x_2 - 2x_3 \geq 2 \quad (2)$

$-x_1 - x_2 - x_3 \geq -7 \quad (3)$

$x_1, x_2, x_3 \geq 0$

$(4, 3, 0) \quad -1$

dual

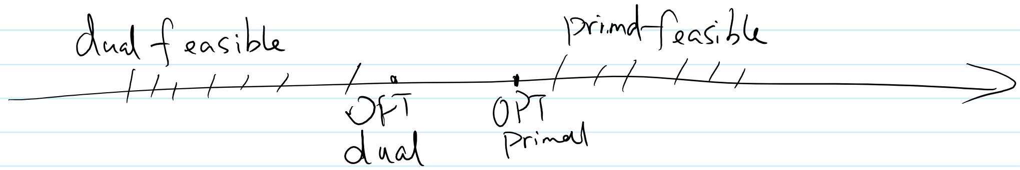
$$\boxed{\begin{aligned} &\max \; y_1 + 2y_2 - 7y_3 \\ &y_1 - y_3 \leq 2 \\ &-y_1 + y_2 - y_3 \leq -3 \\ &-2y_2 - y_3 \leq 1 \\ &y_1, y_2, y_3 \geq 0 \end{aligned}}$$

$(2.5, 0, 0.5) \quad -1$

weak duality

- Claim: weak duality

feasible of dual $\le$ OPT of dual $\boxed{\le}$ OPT of primal $\le$ feasible of primal



- Theorem: strong duality OPT of dual = OPT of primal