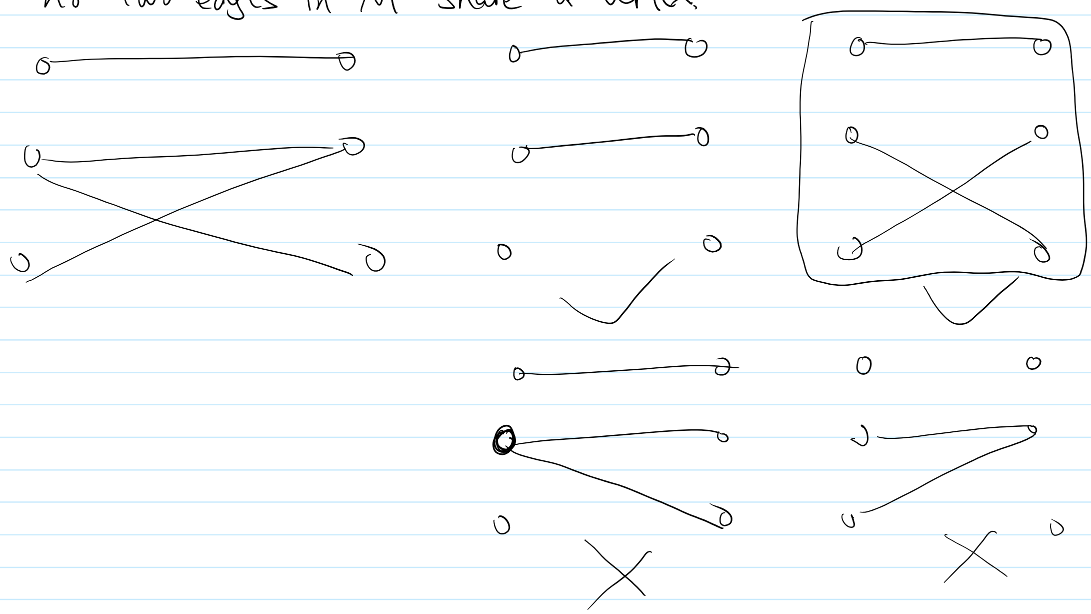
Lecture 17 Bipartite Matching and Max Flow

Monday, March 30, 2020 2:54 PM

- bipartite matching
  - A bipartite graph $G = (U, V, E)$, $E \subseteq U \times V$ (every edge connects a vertex in U and a vertex in V)
  - matching M is a subset of edges $M \subseteq E$ such that no two edges in M share a vertex.



- using linear program for matching
  - for every $(i,j) \in E$, have a variable $\underline{X_{ij}} = \begin{cases} 0 & (i,j) \notin M \\ 1 & \underline{(i,j) \in M} \end{cases}$

every course can use at most 1 classroom

$$\underline{\forall i} \quad \boxed{\sum_{(i,j)\in E} X_{ij} \le 1}$$

every classroom can only be assigned to 1 course

$$\forall j \quad \boxed{\sum_{(i,j)\in E} X_{ij} \le 1}$$

ideally $X_{i,j} = 0, 1$

$$\forall (i,j) \in E \quad 0 \le X_{i,j} \le 1$$

$$\boxed{X_{i,j} \ge 0}$$

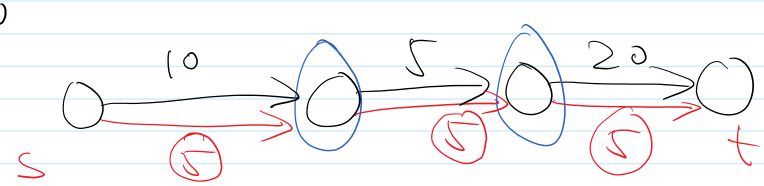
- objective $\max \underbrace{\sum_{(i,j)\in E} X_{i,j}}_{\text{\# edges in matching}}$

- fractional solution vs. $\underbrace{\text{integral solutions}}_{\text{every variable is either 0 or 1}}$ $0 \le X_{i,j} \le 1$

$X_{i,j} = \frac{1}{2}$

- for bipartite matching (max flow), there are always optimal integral solutions. and LP solvers can find integral solution.

- Max Flow



variables for every $(i,j) \in E$, $x_{i,j}$ = amount of flow on edge (i,j)

constraints ① for every edge $\underline{x_{i,j} \leq c_{i,j}} \times \lambda_{i,j}$ $\lambda_{i,j} \geq 0$

② flow conservation

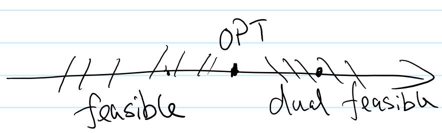
for every $i \neq s,t$

$$\boxed{\underbrace{\sum_{(i,j)\in E} x_{i,j}}_{\text{outgoing flow}} - \underbrace{\sum_{(k,i)\in E} x_{k,i}}_{\text{incoming flow}} = 0} \times y_i$$

y_i can be anything

③ flow direction for $(i,j) \in E$ $x_{i,j} \geq 0$

objective $\boxed{\max \underbrace{\sum_{(i,t)\in E} x_{i,t}}_{\text{flow into } t}} = \max \underbrace{\sum_{(s,i)\in E} x_{s,i}}_{\text{flow out of } s}$



(assuming no edges into s or out of t)

- taking dual

for LHS of (*)	for objective
- coefficient for $\underline{x_{i,j}}$ $(i,j \neq s,t)$	
$\lambda_{i,j} + y_i - y_j$	≥ 0
- coefficient for $\underline{x_{s,j}}$	
$\lambda_{s,j} - y_j$	≥ 0
- coefficient for $\underline{x_{i,t}}$	
$\lambda_{i,t} + y_i$	≥ 1

$$(*) \quad \sum_{(i,j)\neq(s,t)} (\lambda_{i,j} + y_i - y_j) x_{i,j} + \sum_{(s,j)\in E} (\lambda_{s,j} - y_j) x_{s,j}$$

$$+ \sum (\lambda_{i,t} + y_i) x_{i,t} \leq \left(\sum_{(i,j)\in E} c_{i,j} \lambda_{i,j} \right)$$

(*) $\sum_{(i,j)\neq(s,t)} (\lambda_{i,j}(y_i - y_j)) x_{i,j} \quad (i,j)\in E$

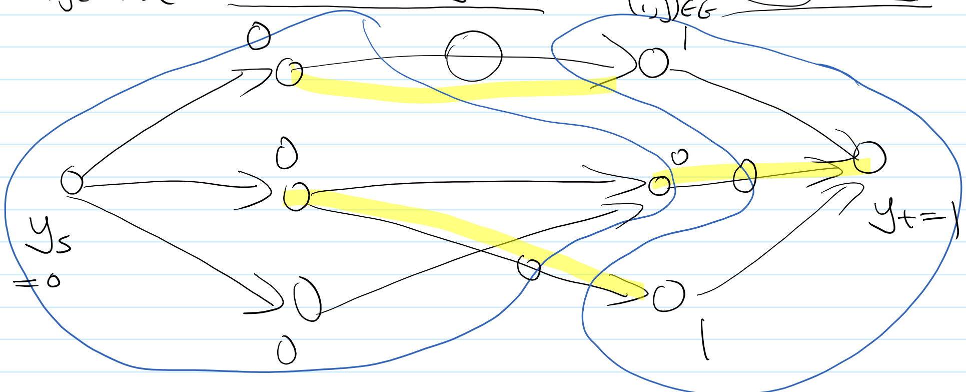
$$+ \sum_{(i,t)\in E} (\lambda_{i,t} + y_i) x_{i,t} \le \sum_{(i,j)\in E} c_{i,j}\lambda_{i,j}$$

want: objective $\le$ LHS of (*)

- simplify : $y_s = 0$, $y_t = 1$

$$\boxed{\begin{array}{l} \forall (i,j)\in E \quad \lambda_{i,j} + y_i - y_j \ge 0 \\ \lambda_{i,j} \ge 0 \end{array}}$$

- objective $\underline{\min \text{RHS of } (*)} = \underline{\sum_{(i,j)\in E} c_{i,j}\lambda_{i,j}}$



$$\lambda_{i,j} = \max(\underline{y_j} - \underline{y_i}, 0)$$

$$\underline{\lambda_{i,j}} = \begin{cases} 0 & y_j = 0 \text{ or } y_i = 1 \\ 1 & y_j = 1 \text{ and } y_i = 0 \end{cases}$$