Lecture 19 Randomized Algorithms 1

Monday, April 6, 2020 2:59 PM

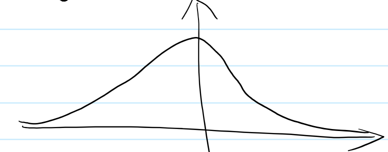
- Probability recap.
  - random variable: a random variable is a variable whose value depends on some random phenomenon.

    use $X, Y, Z \ldots$ to denote random variables

    - example: coin $X = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$

      die $Y = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases}$ w.p. $\frac{1}{6}$

      discrete



      continuous $X \underset{\text{uniform}}{\sim} [0,1]$ $\quad X \sim N(0,1)$

  - event: a set of possible outcomes for a random variable

    $\underline{X=1}$ $\quad Y=3$ $\quad Y \geq 4$ $\quad Y \in \{1,2,4,6\}$

- joint probability
  - probability that multiple events <u>all happen</u>

    $X, Y \sim$ dice

    $$\Pr[X=1, Y=2] = \frac{1}{36}$$

  - independence: outcomes of two random variables do not influence each other.

    $$\Pr[\underline{X=1, Y=2}] = \Pr[X=1]\Pr[Y=2] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

- conditioning
  - probability for X after outcome of Y has been revealed
  - $\underline{\Pr[X=i \mid Y=j]}$
  - $$\Pr[X=i \mid Y=j] = \frac{\Pr[X=i, Y=j]}{\Pr[Y=j]}$$

$$\boxed{\underbrace{Pr[X=i \mid Y=j]}_{\text{Bayes rule}} = \frac{Pr[X=i] \cdot Pr[Y=j \mid X=i]}{Pr[Y=j]} = \frac{Pr[X=i]\, Pr[Y=j \mid X=i]}{\sum_{i \in \Omega_X} Pr[Y=j \mid X=i]\, Pr[X=i]}}$$

$$\boxed{Pr[Y=j] = \sum_{i \in \Omega_X} Pr[Y=j, X=i] = \sum_{i \in \Omega_X} Pr[Y=j \mid X=i]\, Pr[X=i]}$$

- Expectation: "average" value of a random variable

$$\mathbb{E}[X] = \sum_{i \in \Omega_X} Pr[X=i] \cdot i$$

- linearity of expectation $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

- $X \sim$ die $\quad \underline{Y=X} \quad Z = X+Y = \begin{cases} 2 \\ 4 \\ 6 \\ 8 \\ 10 \\ 12 \end{cases} \frac{1}{6}$

$$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = 7$$

$$Y = 7-X \quad Z = X+Y = 7$$

$$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = 7$$

- conditional expectation

$$\mathbb{E}[X \mid Y=j] = \sum_{i \in \Omega_X} Pr[X=i \mid Y=j] \cdot i$$

- law of total expectations

$$\underline{\mathbb{E}[X] = \sum_{j \in \Omega_Y} \mathbb{E}[X \mid Y=j] \cdot Pr[Y=j]}$$

- randomized quicksort.

Let X_n be runtime of quicksort on array of n numbers.

pivot be the ranking of the pivot number

$[\text{pivot} = i]$ event that the pivot number is the i-th smallest number in the array

$$E[X_n] = \sum_{i=1}^{n} E[X_n \mid \text{pivot} = i] \Pr[\text{pivot} = i]$$

$$\Pr[\text{pivot} = i] = \frac{1}{n}$$

$$X_n \mid \text{pivot} = i = \underbrace{n}_{\substack{\text{time to} \\ \text{partition} \\ \text{the array}}} + \underbrace{X_{i-1}}_{\substack{\text{time to} \\ \text{run Quicksort} \\ \text{on left partition}}} + \underbrace{X_{n-i}}_{\text{right partition}}$$

$$E[X_n] = \frac{1}{n} \sum_{i=1}^{n} \left(n + E[X_{i-1}] + E[X_{n-i}]\right)$$

- Proof by induction $E[X_n] \le C \cdot \boxed{n \log_2 n}$

base case: $n = 1$ $E[X_1] = 0$ $E[X_0] = 0$

- induction: assume $E[X_k] \le C \cdot k \cdot \log_2 k$ for all $k < n$

$$E[X_n] = \frac{1}{n} \sum_{i=1}^{n} \left(n + E[X_{i-1}] + E[X_{n-i}]\right)$$

$$= n + \frac{1}{n} \sum_{i=1}^{n} \left(\underbrace{E[X_{i-1}]}_{\cancel{X_0}, X_1, \dots, X_{n-1}} + \underbrace{E[X_{n-i}]}_{X_{n-1}, X_{n-2}, \dots X_1, \cancel{X_0}}\right)$$

$$= n + \frac{2}{n} \boxed{\sum_{i=1}^{n-1} E[X_i]}$$

$$\le n + \frac{2}{n} \sum_{i=1}^{n-1} C \cdot i \cdot \log_2 i$$

$1 \le i \le \frac{n}{2}$ $\log_2 i \le \log_2 \frac{n}{2} = (\log_2 n - 1)$

$\frac{n}{2} < i < n$ $\log_2 i \le \log_2 n$

$$\le n + \frac{2}{n} \sum_{i=1}^{\frac{n}{2}} C \cdot i \cdot (\log_2 n - 1) + \frac{2}{n} \sum_{i=\frac{n}{2}+1}^{n-1} C \cdot i \log_2 n$$

$$\leq n + \frac{2}{n}\sum_{i=1}^{\frac{n}{2}} C \cdot i \cdot (\log_2 n - 1) + \frac{2}{n}\sum_{i=\frac{n}{2}+1} C \cdot i \log_2 n$$

$$= n + \underbrace{\frac{2}{n}\sum_{i=1}^{n-1} C \cdot i \cdot \log_2 n}_{\frac{2}{n} C \frac{n(n-1)}{2} \log_2 n} - \frac{2}{n}\sum_{i=1}^{\frac{n}{2}} C \cdot i$$

$C n \log_2 n$

$$\leq \boxed{n + C n \log_2 n} - \frac{2}{n} \cdot C \cdot \frac{\frac{n}{2} \cdot \frac{n}{2}}{2}$$

$$= C n \log_2 n + n - C \cdot \frac{n}{4}$$

$$\leq \underline{C n \log_2 n} \quad (\text{as long as } C \geq 4)$$

by induction, we know $E[X_n] \leq 4 n \log_2 n$. □