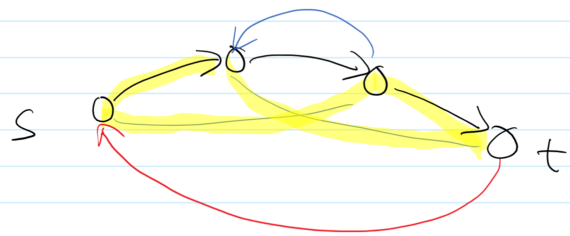
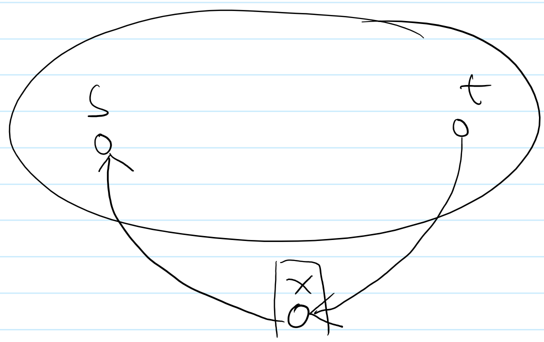
Lecture 23 Classical NP-hard Problems

Monday, April 20, 2020 3:00 PM

- Hamiltonian Path to TSP cycle
  - similarity: visit all vertices
  - differences 1. unweighted graph vs. weighted graph → set all edges to have weight 1
    2. path vs. cycle.



Q: how to ensure the edge from t to s is selected?



- ① given a Hamiltonian Path instance (with n vertices), add a vertex x, add edges (t, x), (x, s), set all edges to have weight 1, set $L = n+1$, this will be the TSP cycle instance
- ② for any hamiltonian path P in original graph
  P + (t, x) + (x, s) form a TSP cycle of length n+1
- ③ For any cycle of length n+1 that visits every vertex, it must visit every vertex exactly once
  cut the cycle when it visits vertex x
  (the only way to visit x is by (t, x), (x, s))
  we get a path from s to t that visits every

vertex in original graph exactly once. $\square$

- 3-SAT to quadratic programming
  - $(x_1 \vee x_2 \vee \overline{x_5}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4}) \wedge \cdots$

    $x_i = \begin{cases} 0 \\ 1 \end{cases}$
  - clause: or of 3 "literals"
  - literal: x or $\overline{x}$
  - formula: and of many clauses
  - goal: is there an $\boxed{\text{assignment to the variables}}$ such that $\boxed{\text{all clauses are satisfied}}$.

- quadratic programming
  - variables $x_1, \cdots, x_n \in \mathbb{R}$
  - constraints: $x_1^2 \leq 3 \quad x_1^2 + 2x_1x_2 - x_2^2 \geq 5$

    $x_1^2 - 2x_2 \geq x_3^2 - x_2x_3$
  - goal: is there an $\boxed{\text{assignment to the variables}}$ such that $\boxed{\text{all constraints are satisfied}}$.

- reduction
  - 1. boolean var. vs. real var.
  - 2: 3-SAT clause vs. quadratic constraint.
  - Q: can we somehow constrain a real var to have only two values?

$$\underline{x_1^2 - x_1 = 0} \qquad x_1 = 0 \text{ or } 1$$

$$(x_1 \vee \underline{x_2} \vee \underline{\overline{x_5}}) \qquad \underbrace{\underline{x_1} + \underline{x_2} + (1 - x_5)}_{\text{\# of literals satisfied in clause}} \geq 1$$

Given a 3-SAT instance, for every variable x_i in 3-SAT create a variable y_i in Q.P., add constraint $y_i^2 - y_i = 0$

for every clause in 3-SAT $(x_a \vee x_b \vee \overline{x_c})$ create a constraint $y_a + y_b + (1 - y_c) \geq 1$

- TRIPARTITE MATCHING to SUBSET VECTORS
  - Similarity: T.M.: <u>select</u> hyperedges, S.V.: <u>select</u> vectors
  - difference: 1. hyper-edges vs. vectors
    2. exactly n hyperedges vs. select any number of vectors.

  - idea: encode hyperedges as vectors.

    $3n$ vertices, a hyper edge: 3 of the $3n$ vertices

    (u_1, v_2, w_5)

    Create vectors of $3n$ dimensions, each dimension corresponds to a vertex

$$
\begin{array}{lccccccccccccccccl}
 & u_1 & u_2 & \cdots & u_n & v_1 & v_2 & \cdots & v_n & w_1 & w_2 & \cdots & w_5 & \cdots & w_n & \\
(u_1, v_2, w_5) \rightarrow & (1 & 0 & \cdots & 0 & 0 & 1 & 0\,0\,0\,0\,0 & 0 & 0 & \cdots & & 1 & \cdots & 0) & 3 \\
(u_2, v_1, w_n) \rightarrow & (0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & & & & \cdots & 0\;1) & 3
\end{array}
$$

in T.M. select n hyper-edges s.t. every vertex is adjacent to exactly 1 hyper-edge.

Sum of these hyper-edges should be equal to

$\underline{u} = (1\ 1\ 1\ \cdots\ 1)$ $3n$

SUBSET VECTORS to SUBSET SUM

Key observation: a number can be viewed as a vector if we take its base $\underline{B}$ representation.

$\boxed{0 \cdots B-1}\ \boxed{0 \cdots B-1}\ \boxed{0 \cdots B-1}\ \boxed{0 \cdots B-1}\ \boxed{0 \cdots B-1}\ \boxed{0 \cdots B-1}$

want: sum of numbers behave like sum of vectors.

true when there is no carry operation

$1001 + 110 = 1111$

(1, 0, 0, 1) + (0, 1, 1, 0) = (1, 1, 1, 1)

1009 + 101 = 11 0 0

(1, 0, 0, 9) + (0, 1, 0, 1) = (1, 1, 0, 10)

Choose B to be very large, so when we take sum, there will be no carry operation.

- SUBSET SUM, DP runs in time $O(n\,m)$

input length = $\boxed{n \lceil \log_2 m \rceil}$

nm is not a polynomial of $\underline{n \lceil \log_2 m \rceil}$