Lecture 24 Amortized Analysis

Wednesday, April 22, 2020 3:00 PM

- Dynamic array

[ ]

capacity = 1 = c

length = 0 = l

add(1) [1] c = l = 1

add(2) [1|2] c = l = 2

add(3) [1|2|3|] c = 4, l = 3

add(4) [1|2|3|4] c = l = 4

K=2 add(5) [1|2|3|4|5| | |] c = 8 l = 5

add(6) add(7) add(8)

light operation

cost 1

heavy operation

$i = 2^k + 1$

cost $2^k + 1$

before: $c = 2^k$ $l = 2^k$

after: $c = 2^{k+1}$ $l = 2^k + 1$

- aggregate method.

$$\text{amortized cost of add} = \frac{\sum_{i=1}^{n} \text{cost for i-th add operation}}{n}$$

$$n \times \text{amortized cost} = \underbrace{\sum_{k: 2^k+1 \le n} (2^k + 1)}_{\text{total cost for heavy op}} + \underbrace{\sum_{\substack{1 \le i \le n \\ i \ne 2^k+1}} 1}_{\text{total cost for light op.}}$$

assume $n = 2^t + 1$

$$= \sum_{k=0}^{t} (2^k + 1) + \sum_{\substack{1 \le i \le n \\ i \ne 2^k+1}} 1$$

$$= \underbrace{\sum_{k=0}^{t} 2^k}_{2^{t+1} - 1 = 2n-3} + \underbrace{\sum_{1 \le i \le n} 1}_{n}$$

$$= 3n - 3$$

$n \times$ amortized cost $= 3n - 3$

$$\text{amortized cost} = \frac{3n-3}{n} \le 3 = O(1).$$

$$\text{amortized cost} = \frac{3n-3}{n} \le 3 = \underline{O(1)}.$$

- accounting (charging) method
  - between two heavy operation

$$2^k+1 \qquad \underbrace{2^k+2 \quad 2^k+3 \quad \cdots \quad 2^{k+1}}_{\text{light operations}} \qquad 2^{k+1}+1$$

heavy

$c=2^{k+1}$
$l=2^k+1$

light operations
$2^{k+1}-(2^k+1)$
$=2^k-1$

$c=2^{k+1}$
$l=2^{k+1}$

new heavy operation

$\boxed{2^{k+1}+1}$

idea: to pay for the heavy operation $i=2^{k+1}+1$
charge some of that to the light operations in between

$$\begin{array}{ll} \text{Cost of heavy} \longrightarrow & \dfrac{2^{k+1}+1}{2^k-1} \approx 2 \\ \text{\# of light op. charging to} \longrightarrow & \end{array}$$

charge 2 for every light operation

new cost for light operation: $1 + \underset{\text{charging}}{2} = 3$

new cost for heavy operation: $2^{k+1}+1 - \underbrace{2(2^k-1)}_{\text{charging}} = \underline{3}$

$\Rightarrow$ after charging, every operation has cost $\underline{3}$,
amortized cost $= O(1)$

Potential argument.

Φ: function on the $\underline{\text{state of the system}}$
c: capacity
l = length

$$\Phi(c,l) = \underline{2l-c}$$

light operation $\quad T=1 \quad (c,l) \longrightarrow (c,l+1)$

$$A = T - (2l-c) + (2(l+1)-c) \qquad (\text{potential} \underline{+2})$$
$$= 3$$

$= 3$

heavy operation $\quad i = 2^k + 1 \quad T = 2^k + 1$

$$(2^k, 2^k) \longrightarrow (2^{k+1}, 2^k + 1)$$

$$A = T - (2 \times 2^k - 2^k) + (2 \times (2^k + 1) - 2^{k+1})$$

(potential $-(2^k - 2)$)

$$= 2^k + 1 - 2^k + 2 = 3$$