Lecture 2: Divide and Conquer I

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January 13, 2020

1 Analyzing Running Time

We will use merge sort to demonstrate how to analyze the running time of a divide-and-conquer algorithm.

1.1 The Algorithm

```
MergeSort(a[]):
0) base case
1) split a[] into b[] and c[]
2) MergeSort(b[]), MergeSort(c[])
3) Merge(b[], c[])
```

The running time (or cost) of merge sort is consists of *merge cost* (the cost of step 0, 1 and 3) and *recursion cost* (the cost of step 2). The merge cost can be analyzed directly.

1.2 The Recurrence Relation $T(n)$

Let $T(n)$ be the running time of the algorithm for an input of size n. We will analyze the merge cost, which will be a function of n, as well as the recursion cost, which will be written as $T(k)$ for some $k < n$. From there, $T(n)$ is simply the sum of merge cost and recursion cost.

For merge sort, the merge cost is $O(n)$, as we go through and combine two sorted sublists in linear time. The recursion cost is $2T(n/2)$, since there are 2 recursive calls, and the input size for each call is $n/2$. Therefore, we have $T(n) = 2T(n/2) + O(n)$. Since the base case is a list of size 1 that does not need to be sorted, i.e. $T(1) = 0$, we can be more precise and write $T(n) = 2T(n/2) + n$.

1.3 The Analysis

So how do we solve the recurrence relation $T(n)$? There are 2 methods in general to upper-bound the running time.

1.3.1 Guess-and-Verify

A guess: $T(n) \le cn\log_2 n$. We will verify this guess by proving $T(n) \le cn\log_2 n$ for some c, by strong induction:

Proof. Induction hypothesis: $T(n) \le cn\log_2 n$ for some c.

Base case: when $n = 1$, $T(1) = 0 \le c \cdot 1\log_2 1$ is true for every c.

Induction step: suppose IH is true for all $k < n$. We will show that IH is also true for n.

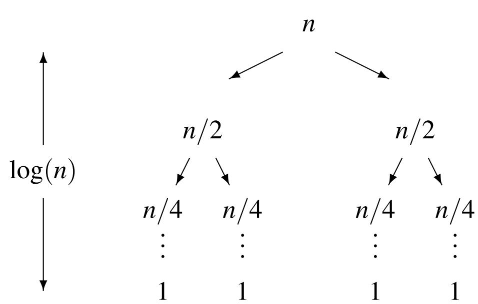
$$\begin{aligned} T(n) &= 2T(n/2) + n \text{ (by recurrence relation)} \\ &\le 2(c\frac{n}{2}\log_2 \frac{n}{2}) + n \text{ (by IH)} \\ &= 2c\frac{n}{2}(\log_2 n - 1) + n \\ &= cn\log_2 n - cn + n \end{aligned}$$

And $T(n) \le cn\log_2 n - cn + n \le cn\log_2 n$ is true whenever $c \ge 1$. □

Therefore, $T(n) \le cn\log_2 n$ for some c. Hence $T(n) = O(n\log n)$

1.3.2 Recursion Tree

We will draw a tree of all recursive calls: each node represents a recursive call; each edge represents one call calling another; each leaf is a base case. From there, $T(n)$ can be calculated by summing the merge cost over every node in the recursion tree. For merge sort, we have:



Below is a summary of this tree.

Depth	Number of nodes	Problem size (each node)	Total problem size
0	1	n	n
1	2	$n/2$	n
2	4	$n/4$	n
⋮			
$\log_2(n) - 1$	$2^{\log_2(n)} = n$	1	n

We are now ready to sum the merge cost over every node in the recursion tree, which is simply

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_2 n-1} \text{merge cost for level } i \\
&= \sum_{i=0}^{\log_2 n-1} n \\
&= n \log_2 n
\end{aligned}
$$

One way to interpret the recursion tree method is that we are substituting in the expression for lower levels. That is:

$$
\begin{aligned}
T(n) &= 2(T/2) + n \\
&= 4T(n/4) + 2\frac{n}{2} + n \\
&= 8T(n/8) + 4\frac{n}{4} + 2\frac{n}{2} + n \\
&\cdots
\end{aligned}
$$

From right to left, we can see that we are essentially summing the merge cost for layer 0, layer 1, layer 2, etc. Hence, we are finding

$$
\sum_{i=0}^{\text{\# layers}} \text{merge cost for layer } i
$$