Lecture 8 Greedy Algorithms 2

Wednesday, February 5, 2020 2:12 PM

- Fractional Knapsack

$W = 5$ 3 items $w_1 = 1$ $v_1 = 2$ $v_1/w_1 = 2$

$w_2 = 2$ $v_2 = 3$ $v_2/w_2 = 1.5$

$w_3 = 3$ $v_3 = 4$ $v_3/w_3 = \frac{4}{3}$

solution for fractional Knapsack.

algorithm: select the item with max v_i/w_i

put this item into the Knapsack until

① run out of this item $P_i = 1$

② run out of capacity

repeat until no more items to add.

$P_1 = 1$ remaining capacity $W = 4$

$P_2 = 1$ remaining capacity $W = 2$

$P_3 = \frac{2}{3}$

total value $= 1 \times 2 + 1 \times 3 + \frac{2}{3} \times 4 = \frac{23}{3}$

running time: $O(n \log n)$

- Proof of correctness:

assume towards contradiction that ALG is not optimal,

there must exist an instance of fractional Knapsack

n items, $\{(w_i, v_i)\}_{i=1,2,\dots,n}$, capacity W

without loss of generality $\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \dots \geq \frac{v_n}{w_n}$

further observe if for two items $\frac{v_i}{w_i} = \frac{v_j}{w_j}$, can merge these two items into one item with value $v_i + v_j$, weight $w_i + w_j$

$$\underbrace{v_1 = 2 \quad w_1 = 1 \qquad v_2 = 4 \quad w_2 = 2}$$

$$V_1 = 2 \quad \underbrace{w_1 = 1 \qquad V_2 = 4}_{V = 6 \quad W = 3} \quad w_2 = 2$$

after merging all items with same ratio, can assume

$$\frac{V_1}{W_1} > \frac{V_2}{W_2} > \frac{V_3}{W_3} > \cdots > \frac{V_m}{W_m}$$

let alg's solution be $(P_1, P_2, \cdots, P_m)$

let opt be $(q_1, q_2, \cdots, q_m)$

if alg is incorrect, $\underbrace{\sum_{i=1}^{m} P_i V_i}_{\text{value of ALG}} < \underbrace{\sum_{i=1}^{m} q_i V_i}_{\text{value of OPT}}$

let l be the first coordinate where

$$P_l \neq q_l$$

by design of alg, we know $\underline{P_l} > \underline{q_l}$

$$\sum_{i=1}^{l} P_i V_i > \sum_{i=1}^{l} q_i V_i$$

there must be a coordinate $h > l$ s.t. $q_h > P_h \geq 0$

(idea: swap item h with item l)

let $\underline{\varepsilon}$ be a small enough number

total weight $+ \underline{W_l \cdot \varepsilon}$ $\quad q_l' \longleftarrow q_l + \varepsilon$ (putting more item l)

total weight $- W_l \cdot \varepsilon$ $\quad q_h' \longleftarrow q_h - \frac{W_l \cdot \varepsilon}{W_h}$ (putting less item h)

let $q_j' = q_j$ for $j \neq l, h$

for solution q' $\quad \sum_{i=1}^{m} q_i' W_i = \sum_{i=1}^{m} q_i W_i$

$$\sum_{i=1}^{m} q_i' V_i = \sum_{i=1}^{m} q_i V_i + \underbrace{\varepsilon V_l}_{\text{extra value from item } l} - \underbrace{\frac{W_l \cdot \varepsilon}{W_h} \cdot V_h}_{\text{value to lose from item } h}$$

from item l — value to lose from item h

by assumption $\frac{v_l}{w_l} > \frac{v_h}{w_h}$

$$\varepsilon v_l > \frac{w_l}{w_h} \cdot v_h \cdot \varepsilon$$

$$> \sum_{i=1}^{m} q_i v_i$$

value of q' is even better than the value of OPT

this is a contradiction.

therefore alg is optimal. $\square$

- Horn-SAT
  - different clauses

① $x_1 \wedge x_4 \Rightarrow x_2$

x_1	x_4	x_2	
T	T	T	✓
T	T	F	✗
T	F	T	✓
T	F	F	✓
...	...	...	✓

② x_1

x_1	
T	✓
F	✗

③ $\overline{x_1} \vee \overline{x_3}$

x_1	x_3	
T	T	✗
T	F	✓
F	T	✓
F	F	✓