Linear Programming and Game Theory

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What are Linear Programs?

- Linear programs are ***constrained optimization problems***
- Constrained optimization problems ask us to maximize or minimize a function subject to mathematical constraints on the variables
  - Convex programs have convex objective functions and convex constraints
  - Linear programs (special case of convex programs) have linear objective functions and linear constraints
- LPs = generic language for wide range problems
- LP solvers = widely available hammers
- Entire classes and vast expertise invested in making problems look like nails

Real-World Applications

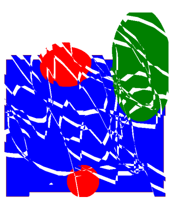
- Railroads – freight car allocation
- Agriculture – optimal mix of crops to plant
- Warfare – logistics, optimal mix of defensive assets, allocation of resources (LP techniques influenced by WWII problems)

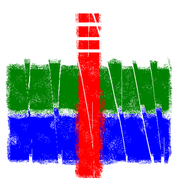
- Networking – capacity management
- Microchips – Optimization of component placement

Photo: Public Domain, https://commons.wikimedia.org/w/index.php?curid=17040973

Linear programs: example

- Make reproductions of 2 paintings





- Painting 1:
  - Sells for $30
  - Requires 4 units of blue, 1 green, 1 red
- Painting 2
  - Sells for $20
  - Requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Solving the linear program graphically

maximize 3x + 2y

subject to

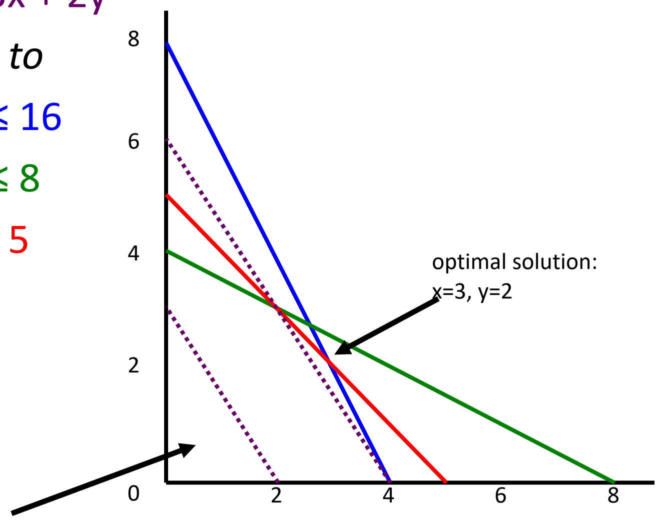
4x + 2y ≤ 16

x + 2y ≤ 8

x + y ≤ 5

x ≥ 0

y ≥ 0



Feasible region = region not violating constraints

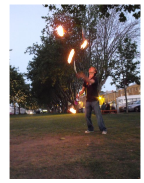
Linear Programs (max formulation)

$$\text{maximize}: c^T x$$
$$\text{subject to}: \mathbf{A}x \le b$$
$$: x \ge 0$$

- Note: min formulation also possible
  - Min: $c^T x$
  - Subject to: Ax≥b
- Some use equality as the canonical representation (introducing slack variables)
- LP tricks
  - Multiply by -1 to reverse inequalities
  - Can easily introduce equality constraints

Solving LPs in Practice

- Use commercial products like cplex or gurobi (there is even an Excel plug-in)



- Don’t implement LP solver yourself!
- Do not use Matlab’s linprog for anything other than small problems. Really. No – REALLY!
- LP Solvers run in (weakly) polynomial time

Photo taken by Liane Moeller - Chris Barnes, Public Domain, https://commons.wikimedia.org/w/index.php?curid=9016956

What is Game Theory? I

- Very general mathematical framework to study situations where multiple agents interact, including:
  - Popular notions of games
  - Everything up to and including multistep, multiagent, simultaneous move, partial information games
  - Example Duke CS research: Aiming sensors to catch hiding enemies, assigning guards to posts
  - Can even include negotiating, posturing and uncertainty about the players and game itself

- von Neumann and Morgenstern (1944) was a major launching point for modern game theory
- Nash: Existence of equilibria in general sum games

(wikipedia)

What is game theory? II

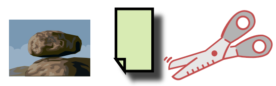
- Study of settings where multiple agents each have
  - Different preferences (utility functions),
  - Different actions
- Each agent’s utility (potentially) depends on all agents’ actions
  - What is optimal for one agent depends on what other agents do
  - Can be circular
- Game theory studies how agents can rationally form beliefs over what other agents will do, and (hence) how agents should act
- Useful for acting and (potentially) predicting behavior of others
- Not necessarily descriptive

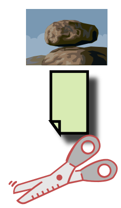
Real World Game Theory Examples

- War
- Auctions
- Animal behavior
- Networking protocols
- Peer to peer networking behavior
- Road traffic
- Mechanism design:
  - Suppose we want people to do X?
  - How to engineer situation so they will act that way?

Rock, Paper, Scissors Zero Sum Formulation

- In zero sum games, one player's loss is other's gain
- Payoff matrix:





	R	P	S
R	0	–1	1
P	1	0	–1
S	–1	1	0

- Minimax solution maximizes worst case outcome

Rock, Paper, Scissors Equations

- R,P,S = probability that we play rock, paper, or scissors respectively (R+P+S = 1)
- U is our expected utility
- Bounding our utility:
  - Opponent rock case: U ≤ P – S
  - Opponent paper case: U ≤ S – R
  - Opponent scissors case: U ≤ R – P
- Want to maximize U subject to constraints
- Solution: (1/3, 1/3, 1/3)

Rock, Paper, Scissors LP Formulation

- Our variables are: $x=[U,R,P,S]^T$
- We want:
  - Maximize U
  - U ≤ P – S
  - U ≤ S – R
  - U ≤ R – P
  - R+P+S = 1
- How do we make this fit:

$$\begin{aligned} &\text{maximize}: c^T x \\ &\text{subject to}: \mathbf{A}x \le b \\ &\qquad\qquad : x \ge 0 \end{aligned}$$

?

Rock Paper Scissors LP Formulation

$$x = [U,R,P,S]^T$$

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix}$$

$$b = [0,0,0,1,-1]^T$$

$$c = [1,0,0,0]^T$$

$$\begin{aligned} &\text{maximize}: c^T x \\ &\text{subject to}: \mathbf{A}x \le b \\ &\qquad\qquad : x \ge 0 \end{aligned}$$

First row of Ax: U – P + S ≤ 0

Rock, Paper, Scissors Solution

- If we feed this LP to an LP solver we get:
  - R=P=S=1/3
  - U=0
- Solution for the other player is:
  - The same…
  - By symmetry
- This is the minimax solution
- This is also an equilibrium
  - No player has an incentive to deviate
  - (Defined more precisely later)

Tangent: Why is RPS Fun?

- OK, it's not…
- Why *might* RPS be fun?
  - Try to exploit non-randomness in your friends
  - Try to be random yourself

Generalizing

- We can solve any two player, simultaneous move, zero sum game with an LP
  - One variable for each of player 1's actions
  - Variables must be a probability distribution (constraints)
  - One constraint for each of player 2's actions (Player 1's utility must be less than or equal to outcome for each player 2 action.)
  - Maximize player 1's utility
- Can solve resulting LP using an LP solver in time that is polynomial in total number of actions

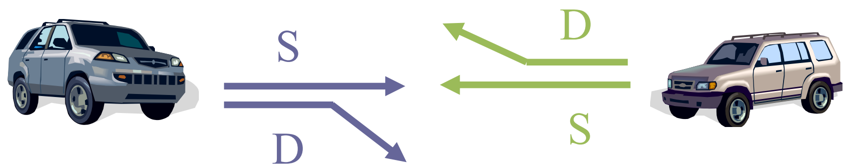
Minimax Solutions in General

- What do we know about minimax solutions?
  - Can a suboptimal opponent trick minimax?
  - When should we abandon minimax?
- Minimax solutions for 2-player zero-sum games can always be found by solving a linear program
- The minimax solutions will also be equilibria (more on that later)
- For general sum games:
  - Minimax does not apply
  - Solutions (equilibria) may not be unique
  - Need to search for equilibria using more computationally intensive methods

General Sum Games

"Chicken"

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die



Source: wikipedia

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

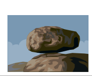
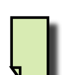
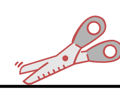
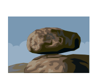
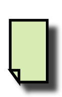
not zero-sum

Reasoning About General Sum Games

- Can’t approach as an optimization problem
- Minimax doesn’t apply
  - Other players’ objectives might be aligned w/ yours
  - Might be partially aligned
- Need a solution concept where each players is “satisfied” WRT his/her objectives

Rock-paper-scissors – Seinfeld variant

MICKEY: All right, rock beats paper!
(Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

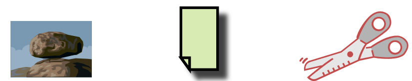
			
	0, 0	1, -1	1, -1
	-1, 1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

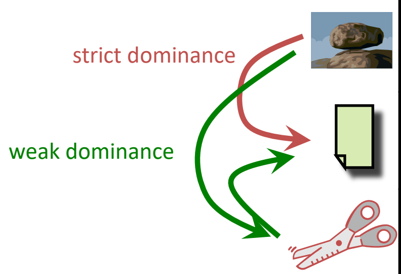
Note: still zero-sum, but useful for understanding a different way of thinking about game solutions.

Dominance

- Player i's strategy s_i strictly dominates s_i' if
  - for any s_{-i}, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- s_i weakly dominates s_i' if
  - for any s_{-i}, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some s_{-i}, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

-i = "the player(s) other than i"





0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

Prisoner's Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

	confess	don't confess
confess	-2, -2	0, -3
don't confess	-3, 0	-1, -1

“Should I buy an SUV?”

purchasing + gas cost


cost: 5


cost: 3

accident cost

cost: 5  cost: 5

cost: 8  cost: 2

cost: 5  cost: 5

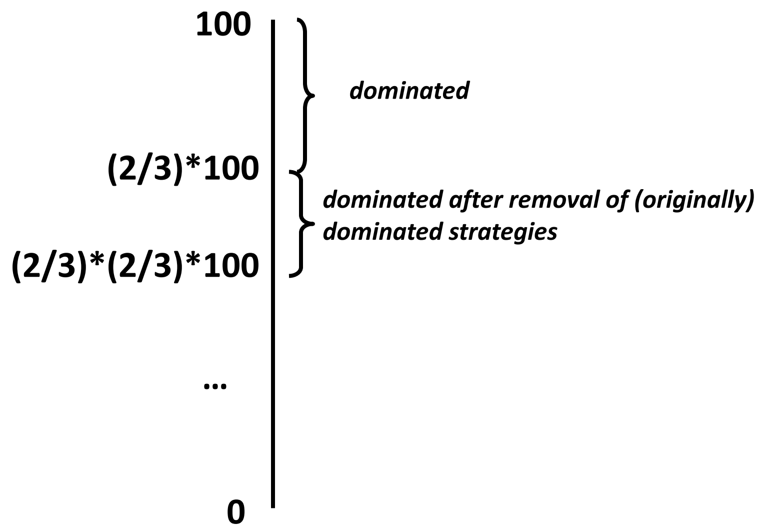


	-10, -10	-7, -11
	-11, -7	-8, -8

“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - 2/3 of average = 33.33
  - A is closest (|50-33.33| = 16.67), so A wins

"2/3 of the average" game revisited



Iterated dominance

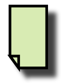
- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld's RPS:

0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

→

0, 0	1, -1
-1, 1	0, 0

Mixed strategies

- Mixed strategy for player i = probability distribution over player i's (pure) strategies
- E.g. 1/3  1/3 , 1/3 
- Example of dominance by a mixed strategy:

1/2	3, 0	0, 0
1/2	0, 0	3, 0
	1, 0	1, 0

Best Responses

- Let A be a matrix of player 1's payoffs
- Let σ_2 be a mixed strategy for player 2
- $A\sigma_2$ = vector of expected payoffs for each strategy for player 1
- Highest entry indicates **best response** for player 1
- Any mixture of ties is also BR, but *can only tie a pure BR*
- Generalizes to >2 players

0, 0	-1, 1
1, -1	-5, -5

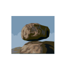
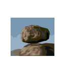
σ_2

Nash equilibrium [Nash 50]

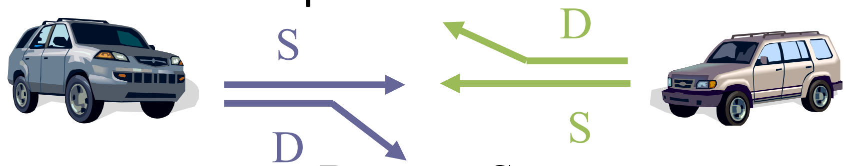

- A vector of strategies (one for each player) = a strategy profile
- Strategy profile ($\sigma_1, \sigma_2, \ldots, \sigma_n$) is a Nash equilibrium if each σ_i is a best response to σ_{-i}
  - That is, for any i, for any σ_i', $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$
- Does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]
- (Note - singular: equilibrium, plural: equilibria)

Equilibrium Strategies vs. Best Responses

- equilibrium strategy -> best response?
- best response -> equilibrium strategy?
- Consider Rock-Paper-Scissors
  - Is (1/3, 1/3, 1/3) a best response to (1/3, 1/3, 1/3)?
  - Is (1, 0, 0) a best response to (1/3, 1/3, 1/3)?
  - Is (1, 0, 0) a strategy for any equilibrium?

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

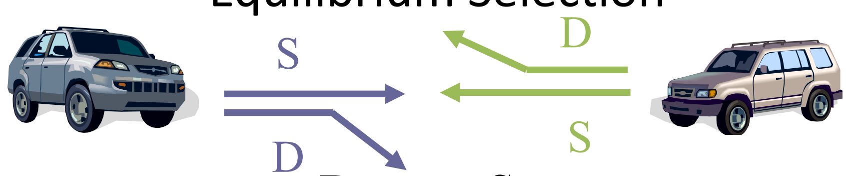
Nash equilibria of "chicken"



	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- (D, S) and (S, D) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria

Equilibrium Selection

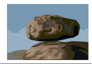
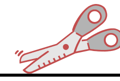
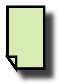


	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- (D, S) and (S, D) are Nash equilibria
- Which do you play?
- What if player 1 assumes (S, D), player 2 assumes (D, S)
- Play is (S, S) = (-5, -5)!!!

- This is the **equilibrium selection** problem

Rock-paper-scissors

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

- Any pure-strategy Nash equilibria?
- It has a mixed-strategy Nash equilibrium:

  Both players put probability 1/3 on each action

Nash equilibria of "chicken"…

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- Is there a Nash equilibrium that uses mixed strategies -- say, where player 1 uses a mixed strategy?
- ***If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses***
- So we need to make player 1 indifferent between D and S
- Player 1's utility for playing D = $-p^c_S$
- Player 1's utility for playing S = $p^c_D - 5p^c_S = 1 - 6p^c_S$
- So we need $-p^c_S = 1 - 6p^c_S$ which means $p^c_S = 1/5$
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player

$-p^c_S$ = probability that column player plays s

Computational Issues

- Zero-sum games - solved efficiently as LP
- General sum games may require exponential time (in # of actions) to find a single equilibrium (no known efficient algorithm and good reasons to suspect that none exists)
- Some better news: Despite bad worst-case complexity, many games can be solved quickly

Game Theory Issues

- How descriptive is game theory?
  - Some evidence that people play equilibria
  - Also, some evidence that people act irrationally
  - If it is computationally intractable to solve for equilibria of large games, seems unlikely that people are doing this
- How reasonable is (basic) game theory?
  - Are payoffs known?
  - Are situations really simultaneous move with no information about how the other player will act?
  - Are situations really single-shot? (repeated games)
  - How is equilibrium selection handled in practice?

Extensions

- Partial information
- Uncertainty about the game parameters, e.g., payoffs (Bayesian games)
- Repeated games: Simple learning algorithms can converge to equilibria in some repeated games
- Multistep games with distributions over next states (game theory + MDPs = stochastic games)
- Multistep + partial information (Partially observable stochastic games)

- Game theory is so general, that it can encompass essentially all aspects of strategic, multiagent behavior, e.g., negotiating, threats, bluffs, coalitions, bribes, etc.

Conclusions

- Game theory tells us how to act in strategic situations – different agents with different goals acting with awareness of other agents
- Zero sum case is relatively easy
- General sum case is computationally hard – some nice results for special cases
- Extensions address some shortcomings/assumptions of basic model but at additional computational cost