Section: Pushdown Automata

Ch. 7 - Pushdown Automata

A DFA = (Q, Σ, δ, q₀, F)

input tape

a a b b a b

tape head

head moves

current state

0 1
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where

- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) - start stack symbol, \( z \in \Gamma \)
- \( F \subseteq Q \) is set of final states.
- \( \delta: Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^* \)
Example of transitions

\[ \delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\[(q, w, u)\]

Description of a Move:

\[(q_1, aw, bx) \vdash (q_2, w, yx)\]

iff

\[\langle q_2, y \rangle \in S(q_1, a, b)\]

Definition Let \(M=\langle Q, \Sigma, \Gamma, \delta, q_0, z, F \rangle\) be a NPDA. \(L(M)=\{w \in \Sigma^* \mid (q_0, w, z)^* \vdash (p, \lambda, u), p \in F, u \in \Gamma^*\}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.
Example: $L = \{a^n b^n | n \geq 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
Another Definition for Language Acceptance

NPDA $M$ accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* | (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}$$
Example: \( L = \{ a^n b^m c^{n+m} | n, m > 0 \} \),
\( \Sigma = \{ a, b, c \} \), \( \Gamma = \{ 0, z \} \)
Examples for you to try on your own: (solutions are at the end of the handout).

- $L = \{a^n b^m | m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
- $L = \{a^n b^{n+m} c^m | n, m > 0\}$, $\Sigma = \{a, b, c\}$
- $L = \{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$
Definition: A PDA

\( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) is deterministic if for every \( q \in Q, \ a \in \Sigma \cup \{\lambda\}, \ b \in \Gamma \)

1. \( \delta(q, a, b) \) contains at most 1 element

2. if \( \delta(q, \lambda, b) \neq \emptyset \) then \( \delta(q, c, b) = \emptyset \) for all \( c \in \Sigma \)

Definition: \( L \) is DCFL iff \( \exists \) DPDA \( M \) s.t. \( L = L(M) \).
Examples:

1. Previous pda for \(\{a^n b^n | n \geq 0\}\) is deterministic?

2. Previous pda for \(\{a^n b^m c^{n+m} | n, m > 0\}\) is deterministic?

3. Previous pda for \(\{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\}\) is deterministic?
Example: \( L = \{ a^n b^m | m > n, m, n > 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;az} q_1 \\
a,a;aa \quad b,a;\lambda \\
q_1 \xrightarrow{b,a;\lambda} q_2 \\
b,z;z \\
q_2 \xrightarrow{b,z;z} q_3 \\
\end{array}
\]

Example: \( L = \{ a^n b^{n+m} c^m | n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \), \( \Gamma = \{ z, a \} \)

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;az} q_1 \\
a,a;aa \\
b,a; \quad b,a; \\
b,z;bz \\
q_1 \xrightarrow{b,z;bz} q_2 \\
\quad \quad \quad c,b; \\
q_2 \xrightarrow{c,b;} q_4 \\
\quad \quad \quad z;z \\
q_4 \xrightarrow{z;z} q_5 \\
q_5 \xrightarrow{z;z} q_4 \\
\end{array}
\]

Example: \( L = \{ a^n b^{2n} | n > 0 \} \), \( \Sigma = \{ a, b \} \)

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;az} q_1 \\
a,a;aaa \\
b,a; \quad b,a; \\
\lambda \quad \lambda \\
q_1 \xrightarrow{b,a; \lambda} q_2 \\
\quad \quad \quad \lambda ;z; \lambda \\
q_2 \xrightarrow{\lambda ;z; \lambda} q_3 \\
\end{array}
\]