Section: Properties of Regular Languages

Example

$$L = \{a^nba^n \mid n > 0\}$$

NOT regular

for $n \leq 10$

Closure Properties

A set is closed over an operation if

$$L_1, L_2 \in \text{class}$$

$$L_1 \text{ op } L_2 = L_3$$

$$\Rightarrow L_3 \in \text{class}$$
L = \{x \mid x \text{ is a positive even integer}\}

L is closed under

addition?  yes
multiplication? yes
subtraction?  no
division?  no

Closure of Regular Languages

Theorem 4.1 If \(L_1\) and \(L_2\) are regular languages, then

\[ L_1 \cup L_2 \]
\[ L_1 \cap L_2 \]
\[ L_1 L_2 \]
\[ \overline{L}_1 \]
\[ L_1^* \]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- final state $\Rightarrow$ not final state
- state not final $\Rightarrow$ final state
- trap state
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'( (q_i, p_i), a) = (q_k, p_e)$ if $\delta_1((q_i, a) = q_k) \in M_1$ and $\delta_2((p_i, a) = p_e) \in M_2$

$F' = \{ (q_i, p_i) \in Q' \mid q_i \in F_1, p_i \in F_2 \}$

Show $\text{wel}(M') \subseteq \text{wel}(L_1 \cap L_2)

\Rightarrow$ closed under intersection
Example:

M1

1 \rightarrow b \rightarrow 2

\rightarrow a

M2

A \rightarrow a \rightarrow B \rightarrow a \rightarrow C

\rightarrow a, b

intersection

\rightarrow a \rightarrow B \rightarrow a \rightarrow IC \rightarrow 2C

\rightarrow a

aaab
Regular languages are closed under

reversal \[ L^R \]
difference \[ L_1 - L_2 \]
right quotient \[ L_1 / L_2 \]
homomorphism \[ h(L) \]
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
L_1 = \{ a^*b^* \cup b^*a^* \} \\
L_2 = \{ b^n \mid n \text{ is even, } n > 0 \} \\
L_1/L_2 = \{ a^*b^* \}
\]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=\langle Q, \Sigma, \delta, q_0, F \rangle$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

$$h(a) = 11$$

$$h(b) = 00$$

$$h(c) = 0$$

$$h(bc) = \text{[Diagram]}$$

$$h(ab^*) = \text{[Diagram]}$$
Questions about regular languages:
L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?
  
  Construct DFA. Test if it accepts w.

• Is L empty?
  
  L = \{ \epsilon, a^i b^j | i \geq 0, j \geq 0 \}
  
  Is there a path from start state to final state?

• Is L infinite?
  
  Does it have a loop?

• Does L₁ = L₂?
  
  Construct L₃ = (L₁ ∪ L₂) \cup (L₁ \cap \overline{L₂})
  
  If L₃ = ø then L₁ = L₂
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

- Yes, union all strings together in regex expr.

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^nb^m | n > 0, m > 0\} = \text{aa\_b\_b}$
- $L_2 = \{a^n b^n | n > 0\}$ not regular
Prove that \( L_2 = \{a^n b^n | n > 0 \} \) is ?

- Proof: Suppose \( L_2 \) is regular.
  \[ \Rightarrow \exists \text{ DFA } M \text{ that recognizes } L_2 \]
  
  \( M \) has a finite no. of states,
  \( k \) states

  Consider a long string in it
  \( a^k b^k \in L_2 \)

  \( \rightarrow \) go thru loop extra time

  the string \( a^k b^k e M \),
  not in language of \( L_2 \)

  Contradiction

  \( \Rightarrow L_2 \) is not regular
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$
|xy| \leq m \\
|y| \geq 1 \\
x y^i z \in L \text{ for all } i \geq 0
$$
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ $L$ satisfies the pumping lemma.
  
  Choose a long string $w$ in $L$, $|w| \geq m$.
  
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  
  The pumping lemma does not hold. Contradiction!
  
  $\Rightarrow$ $L$ is not regular. QED.
Example \( L = \{ a^n c b^n | n > 0 \} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = a c b^m \in L \)

\[ w = xyz \]

\[ a^q c b^m \rightarrow a^q c b^{m+1} \rightarrow a^q c b^m \]

\[ x y z \]

all partitions are of the form \( x = a^k, y = a^j, z = a^m c b^j \)

\[ k + j + m \]
it should be true that $xyzL$ for all $i>0$.

$i=1$ $xyz \in L$.

$i=2$ $xxyz = a^m b^l c^m \notin L$ contradiction

$i=0$ $xz = a^{m-j} c^m \notin L$.

$i=3$ $xxyyz = a^{m+j} c^b \notin L$.

Only have to show one $i$ doesn't work.

Contradiction!

$\Rightarrow L$ is not regular.
Example $L = \{a^n b^{n+s} c^s | n, s > 0 \}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w = a^m b^{m+2} c^m \
  \text{general for all partitions}$

So the partition is:

$w = xyz$

$|y| > 0$

$xy^iz \in L$ for all $i \geq 0$

$xy^2z \notin L$
Example $\Sigma = \{a, b\}$,
$\mathcal{L} = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) \}$

$\mathcal{L}$ is not regular.

- Proof:
  Assume $\mathcal{L}$ is regular.
  $\Rightarrow$ the pumping lemma holds.

Choose $w = b^m a^{m+1} b^m$.

So the partition is:

$w = xyz$

$x = b^i$, $y = b^j$, $z = b^{m-k-j} a$

$i = 0$ does not work

$i > 2 \quad xy^iz = b^{m+j} a \notin \mathcal{L}$

$x, y, z \in \mathcal{L}$

num(b's) $\geq$ num(a's)

Contradiction.
Example $L = \{a^3 b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove L is not regular:

● Proof Outline:
  Assume L is regular.
  Apply closure properties to L and other regular languages, constructing L’ that you know is not regular.
  closure properties ⇒ L’ is regular.
  Contradiction!
  L is not regular. QED.
Example \( L = \{ a^3b^n c^{n-3} | n > 3 \} \)

\( L \) is not regular.

- Proof: (proof by contradiction)
  Assume \( L \) is regular.
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  \( h(a) = a \), \( h(b) = a \), \( h(c) = b \)
  \( h(L) = \)
Example $L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \}$

$L$ is not regular.

- **Proof: (proof by contradiction)**
  
  Assume $L$ is regular.
Example: \( L_1 = \{a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.