Section: Properties of Regular Languages

Example

$L = \{ a^n ba^n | n > 0 \}$

Not regular

for $n \leq 10$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$\Rightarrow L_3 \in \text{class}$
L = \{ x \mid x \text{ is a positive even integer} \}

L is closed under

- addition? yes
- multiplication? yes
- subtraction? no
- division? no

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

\[ L_1 \cup L_2, \quad L_1 \cap L_2, \quad L_1 L_2, \quad \overline{L}_1, \quad L_1^* \]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- Final state $\Rightarrow$ not final state
- State not final $\Rightarrow$ final state
- Trap state
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1) \text{ and } L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'( ((q_i, p_i), a) ) = (q_k, p_e) \text{ iff }$

$\delta_1( (q_i, a) ) = q_k \in M_1 \text{ and }$

$\delta_2( (p_i, a) ) = p_e \in M_2$

$F' = \{ (q_i, p_i) \in Q' \mid q_i \in F_1, p_i \in F_2 \}$

Show $wel(M') \subseteq wel(L_1 \cap L_2)$

$\Rightarrow$ closed under intersection
Example:

\begin{align*}
\text{Example:} & \\
\text{ml} & \\
\text{intersection} & \\
\text{ic} & \\
\text{aaab} & \\
\end{align*}
Regular languages are closed under

reversal \( L^R \)

difference \( L_1 - L_2 \)

right quotient \( L_1 / L_2 \)

homomorphism \( h(L) \)
Right quotient

Def: $L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{a^*b^* \cup b^*a^*\}
L_2 = \{b^n \mid n \text{ is even, } n > 0\}
L_1/L_2 = \{a^*b^*3\}$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \quad \Gamma = \{0, 1\}$$

$$h(a) = 11$$
$$h(b) = 00$$
$$h(c) = 0$$

$$h(bc) = 000$$

$$h(ab^*) = 11(00)^*$$
Questions about regular languages:

L is a regular language.

- Given L, $\Sigma$, $w \in \Sigma^*$, is $w \in L$?
  
  Construct DFA, test it.

- Is L empty?
  
  \[ L = \{a^n b^m | n > 0, m \geq 2n \} \]

  Is there path from start state to final state?

- Is L infinite?

- Does $L_1 = L_2$?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \emptyset$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that \( L_2 = \{ a^n b^n | n > 0 \} \) is regular.

- **Proof:** Suppose \( L_2 \) is regular.
  \[ \Rightarrow \exists \text{ DFA } M \text{ that recognizes } L_2 \]
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \text{ for all } i \geq 0
$$
To Use the Pumping Lemma to prove L is not regular:

- **Proof by Contradiction.**
  Assume L is regular.
  \[ \Rightarrow \text{L satisfies the pumping lemma.} \]
  Choose a long string \( w \) in L, \( |w| \geq m \).
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m, |y| \geq 1 \) and \( xy^i z \in L \ \forall \ i \geq 0 \).
  The pumping lemma does not hold. **Contradiction!**
  \[ \Rightarrow \text{L is not regular. QED.} \]
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
Example $L = \{a^n b^{n+s} c^s | n, s > 0 \}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w = \ldots$
  
  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) \}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

(shown in detail on handout)

$L$ is not regular.
To Use Closure Properties to prove L is not regular:

- **Proof Outline:**
  - Assume L is regular.
  - Apply closure properties to L and other regular languages, constructing L’ that you know is not regular.
  - closure properties $\Rightarrow$ L’ is regular.
  - Contradiction!
  - L is not regular. QED.
Example \( L=\{a^3b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  \( h(a) = a \) \( h(b) = a \) \( h(c) = b \)
  \( h(L) = \)
Example \( L=\{a^n b^m a^m | m \geq 0, n \geq 0\} \)

\( L \) is not regular.

• Proof: (proof by contradiction)
  Assume \( L \) is regular.
Example: $L_1 = \{a^n b^n a^n | n > 0}\$

$L_1$ is not regular.