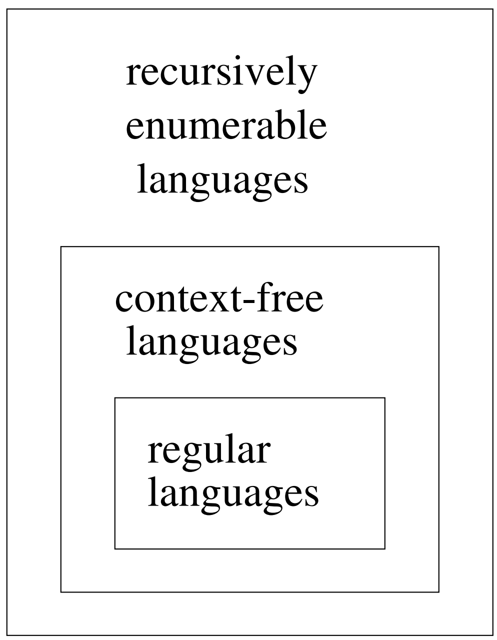
Section: Recursively Enumerable Languages

Definition: A language L is *recursively enumerable* if there exists a TM M such that L=L(M).



Definition: A language L is *recursive* if there exists a TM M such that L=L(M) and M halts on every w$\in \Sigma^+$.

To enumerate all w$\in \Sigma^+$ in a recursive language L:

- **Let M be a TM that recognizes L, L = L(M).**
- **Construct 2-tape TM M'**

  Tape 1 will enumerate the strings in Σ^+

  Tape 2 will enumerate the strings in L.

  - **On tape 1 generate the next string v in Σ^+**
  - **simulate M on v**

    if M accepts v, then write v on tape 2.

To enumerate all w$\in \Sigma^+$ in a recursively enumerable language L:

Repeat forever

- **Generate next string (Suppose k strings have been generated:** $w_1, w_2, ..., w_k$**)**
- **Run M for one step on** w_k

  Run M for two steps on w_{k-1}**.**

  ...

  Run M for k steps on w_1**.**

  If any of the strings are accepted then write them to tape 2.

Theorem Let S be an infinite countable set. Its powerset 2^S is not countable.

Proof - Diagonalization

- **S is countable, so it's elements can be enumerated.**
  S $= \{s_1, s_2, s_3, s_4, s_5, s_6 \ldots\}$
  Example, $\{s_2, s_3, s_5\}$ **represented by**

  Example, set containing every other element from S, starting with s_1 **is** $\{s_1, s_3, s_5, s_7, \ldots\}$ **represented by**

Suppose 2^S countable. Then we can emunerate all its elements: t_1, t_2, ...

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	...
t_1	**0**	**1**	**0**	**1**	**0**	**0**	**1**	**...**
t_2	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**...**
t_3	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**...**
t_4	**1**	**0**	**1**	**0**	**1**	**1**	**0**	**...**
t_5	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**...**
t_6	**1**	**0**	**0**	**1**	**0**	**0**	**1**	**...**
t_7	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**...**
...								

Theorem For any nonempty Σ, there exist languages that are not recursively enumerable.

Proof:

- **A language is a subset of Σ^*.**
  The set of all languages over Σ is

Theorem There exists a recursively enumerable language L such that $\bar{L}$ is not recursively enumerable.

Proof:

- **Let $\Sigma = \{a\}$**
  Enumerate all TM's over Σ:

	a	**aa**	**aaa**	**aaaa**	**aaaaa**	**...**
$\mathbf{L}(M_1)$	**0**	**1**	**1**	**0**	**1**	**...**
$\mathbf{L}(M_2)$	**1**	**0**	**1**	**0**	**1**	**...**
$\mathbf{L}(M_3)$	**0**	**0**	**1**	**1**	**0**	**...**
$\mathbf{L}(M_4)$	**1**	**1**	**0**	**1**	**1**	**...**
$\mathbf{L}(M_5)$	**0**	**0**	**0**	**1**	**0**	**...**
...						

Theorem If languages L and $\bar{L}$ are both RE, then L is recursive.

Proof:

- **$\exists$ M_1 s.t. M_1 can enumerate all elements in L.**

  $\exists$ M_2 s.t. M_2 can enumerate all elements in $\bar{L}$.

  To determine if a string w is in L or not in L

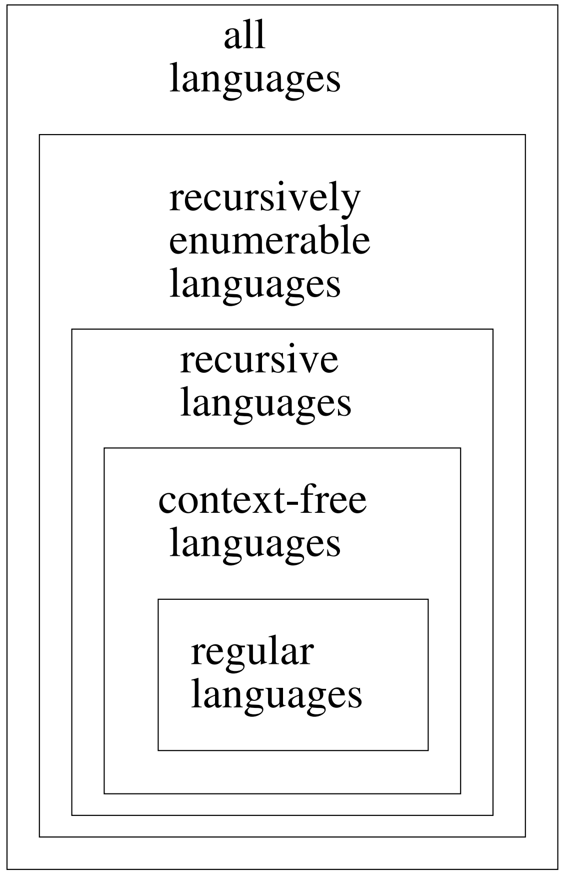
Theorem: If L is recursive, then $\bar{L}$ is recursive.

Proof:

- **L is recursive, then there exists a TM M such that M can determine if w is in L or w is not in L.**

  Construct TM M' that does the following. M' first simulates TM M.

Hierarchy of Languages:



Definition A grammar G=(V,T,S,P) is *unrestricted* **if all productions are of the form**

$$u \to v$$

where $u \in$**(V**$\cup$**T)**$^+$ **and** $v \in$**(V**$\cup$**T)***

Example:

Let G=($\{$**S,A,X**$\}$**,**$\{$**a,b**$\}$**,S,P), P=**

$$\begin{aligned} &\mathbf{S} \to \mathbf{bAaaX} \\ &\mathbf{bAa} \to \mathbf{abA} \\ &\mathbf{AX} \to \lambda \end{aligned}$$

Example Find an unrestricted grammar G s.t. L(G)=$\{a^n b^n c^n | n > 0\}$

G=(V,T,S,P)

V=$\{$S,A,B,D,E,X$\}$

T=$\{$a,b,c$\}$

P=

1) S $\rightarrow$ AX
2) A $\rightarrow$ aAbc
3) A $\rightarrow$ aBbc
4) Bb $\rightarrow$ bB
5) Bc $\rightarrow$ D
6) Dc $\rightarrow$ cD
7) Db $\rightarrow$ bD
8) DX $\rightarrow$ EXc

$$\mathbf{S} \Rightarrow \mathbf{AX} \Rightarrow \mathbf{aAbcX} \Rightarrow \mathbf{aaAbcbcX} \Rightarrow \mathbf{aaaBbcbcbcX}$$

Theorem If G is an unrestricted grammar, then L(G) is recursively enumerable.

Proof:

- List all strings that can be derived in one step.

  List all strings that can be derived in two steps.

Theorem If L is recursively enumerable, then there exists an unrestricted grammar G such that L=L(G).

Proof:

- **L is recursively enumerable.**

  $\Rightarrow$ **there exists a TM M such that L(M)=L.**

  M $= (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

  $q_0 w \overset{*}{\vdash} x_1 q_f x_2$ **for some** $q_f \in$ **F,**

  $x_1, x_2 \in \Gamma^*$

  Construct an unrestricted grammar G s.t. L(G)=L(M).

  $S \overset{*}{\Rightarrow} w$

  Three steps

  1. $S \overset{*}{\Rightarrow} B \ldots B\#xq_f yB \ldots B$

  2. $B \ldots B\#xq_f yB \ldots B \overset{*}{\Rightarrow}$
  $B \ldots B\#q_0 wB \ldots B$

  3. $B \ldots B\#q_0 wB \ldots B \overset{*}{\Rightarrow} w$

Definition A grammar G is *context-sensitive* **if all productions are of the form**

$$x \to y$$

where $x, y \in (V \cup T)^+$ **and** $|x| \leq |y|$

Definition L is context-sensitive (CSL) if there exists a context-sensitive grammar G such that L=L(G) or L=L(G) $\cup \{\lambda\}$**.**

Theorem For every CSL L not including λ, $\exists$ an LBA M s.t. L=L(M).

Theorem If L is accepted by an LBA M, then $\exists$ CSG G s.t. L(M)=L(G).

Theorem Every context-sensitive language L is recursive.

Theorem There exists a recursive language that is not CSL.