Example

\[ L = \{a^n ba^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]

Example

\[ L = \{x \mid x \text{ is a positive even integer} \} \]

L is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

**Theorem 4.1** If \( L_1 \) and \( L_2 \) are regular languages, then

\[ L_1 \cup L_2 \]
\[ L_1 \cap L_2 \]
\[ L_1 L_2 \]
\[ L_1^c \]
\[ L_1^* \]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
⇒ ∃ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
⇒ closed under union

$r_1 r_2$ is r.e. denoting $L_1 L_2$
⇒ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$
⇒ closed under star-closure

complementation:
$L_1$ is reg. lang.
⇒ ∃ DFA $M$ s.t. $L_1 = L(M)$
Construct $M'$ s.t.

intersection:
$L_1$ and $L_2$ are reg. lang.
⇒ ∃ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q'$ =
$\delta'$:
Regular languages are closed under

- reversal   $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$

Right quotient

Def: $L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{ a^*b^* \cup b^*a^* \}$
$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$
$L_1 / L_2 = \quad$ (omitted for brevity)

**Theorem** If $L_1$ and $L_2$ are regular, then $L_1 / L_2$ is regular.

**Proof** (sketch)

exists DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

- Make $i$ the start state (representing $L_i$)

QED.
**Homomorphism**

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h : \Sigma \to \Gamma^*$$

**Example:**

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

$$h(a) = 11$$

$$h(b) = 00$$

$$h(c) = 0$$

$$h(bc) =$$

$$h(ab^*) =$$

**Questions about regular languages:**

L is a regular language.

- Given $L, \Sigma, w \in \Sigma^*$, is $w \in L$?

- Is $L$ empty?

- Is $L$ infinite?

- Does $L_1 = L_2$?
Ch. 4.3 - Identifying Nonregular Languages

If a language L is finite, is L regular?

If L is infinite, is L regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \#
- L_2 = \{a^n b^n | n > 0\}$

Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA M that recognizes $L_2$
**Pumping Lemma:** Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

- $|xy| \leq m$
- $|y| \geq 1$
- $xy^iz \in L \quad \text{for all} \quad i \geq 0$

**Meaning:** Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in $L$.

**To Use the Pumping Lemma to prove $L$ is not regular:**

- **Proof by Contradiction.**
  - Assume $L$ is regular.
  - $\Rightarrow$ $L$ satisfies the pumping lemma.
  - Choose a long string $w$ in $L$, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  - Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  - The pumping lemma does not hold. Contradiction!
  - $\Rightarrow$ $L$ is not regular. QED.

**Example** $L = \{a^ncb^n|n > 0\}$

$L$ is not regular.

- **Proof:**
  - Assume $L$ is regular.
  - $\Rightarrow$ the pumping lemma holds.
  - Choose $w =$ where $m$ is the constant in the pumping lemma. (Note that $w$ must be chosen such that $|w| \geq m$.)
  - The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with $cb^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

  It should be true that $xy^iz \in L$ for all $i \geq 0$. 

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Example $L = \{a^n b^{n+s} c^s \mid n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with the rest of the string $b^{m+s} c^s$.
  
  This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

Example $\Sigma = \{a, b\}$, $L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example \( L = \{a^3b^nc^{n-3} | n > 3\} \)

\( L \) is not regular.

- **Proof:**  Assume \( L \) is regular. \( \Rightarrow \) the pumping lemma holds.

Choose \( w = a^3b^mc^{m-3} \) where \( m \) is the constant in the pumping lemma. There are three ways to partition \( w \) into three parts, \( w = xyz \). 1) \( y \) contains only \( a \)'s 2) \( y \) contains only \( b \)'s and 3) \( y \) contains \( a \)'s and \( b \)'s

We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide \( w \) into three parts s.t. the pumping lemma contraints were true).

**Case 1:** \( (y \) contains only \( a \)'s). Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and \( z \) contains 0 to 2 \( a \)'s concatenated with the rest of the string \( b^m c^{m-3} \), such that there are exactly 3 \( a \)'s. So the partition is:

\[
\begin{align*}
x &= a^k \\
y &= a^j \\
z &= a^{3-k-j} b^m c^{m-3}
\end{align*}
\]

where \( k \geq 0 \), \( j > 0 \), and \( k + j \leq 3 \) for some constants \( k \) and \( j \).

It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

\( xy^2z = (x)(y)(z) = (a^k)(a^j)(a^{3-j-k} b^m c^{m-3}) = a^{3+j} b^m c^{m-3} \notin L \) since \( j > 0 \), there are too many \( a \)'s. Contradiction!

**Case 2:** \( (y \) contains only \( b \)'s) Then \( x \) contains 3 \( a \)'s followed by 0 or more \( b \)'s, \( y \) contains 1 to \( m-3 \) \( b \)'s, and \( z \) contains 3 to \( m-3 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

\[
\begin{align*}
x &= a^3b^k \\
y &= b^j \\
z &= b^{m-j} c^{m-3}
\end{align*}
\]

where \( k \geq 0 \), \( j > 0 \), and \( k + j \leq m-3 \) for some constants \( k \) and \( j \).

It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

\( xy^2z = a^3b^{m-j} c^{m-3} \notin L \) since \( j > 0 \), there are too few \( b \)'s. Contradiction!

**Case 3:** \( (y \) contains \( a \)'s and \( b \)'s) Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and 1 to \( m-3 \) \( b \)'s, \( z \) contains 3 to \( m-1 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

\[
\begin{align*}
x &= a^{3-k} \\
y &= a^kb^j \\
z &= b^{m-j} c^{m-3}
\end{align*}
\]

where \( 3 \geq k > 0 \), and \( m-3 \geq j > 0 \) for some constants \( k \) and \( j \).

It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

\( xy^2z = a^3b^ja^{k}b^{m}c^{m-3} \notin L \) since \( j,k > 0 \), there are \( b \)'s before \( a \)'s. Contradiction!

\( \Rightarrow \) There is no partition of \( w \).

\( \Rightarrow \) \( L \) is not regular! QED.
To Use Closure Properties to prove $L$ is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular.
  
  Contradiction!
  
  $L$ is not regular. QED.

**Example** $L = \{a^{3n}b^{n}c^{n-3} | n > 3\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  
  Assume $L$ is regular.
  
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  
  $h(a) = a$ $h(b) = a$ $h(c) = b$
  
  $h(L) = \ldots$
Example \( L = \{a^n b^m a^m | m \geq 0, n \geq 0\} \)

\( L \) is not regular.

- **Proof**: (proof by contradiction)
  Assume \( L \) is regular.

\[ L_1 = \{a^n b^n a^n | n > 0\} \]

\( L_1 \) is not regular.

- **Proof**:
  Assume \( L_1 \) is regular.
  Goal is to try to construct \( \{a^n b^n | n > 0\} \) which we know is not regular.
  Let \( L_2 = \{a^n\} \). \( L_2 \) is regular.
  By closure under right quotient, \( L_3 = L_1 \setminus L_2 = \{a^n b^n a^p | 0 \leq p \leq n, n > 0\} \) is regular.
  By closure under intersection, \( L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n | n > 0\} \) is regular.
  Contradiction, already proved \( L_4 \) is not regular!
  Thus, \( L_1 \) is not regular. QED.