CompSci 270
Informed Search

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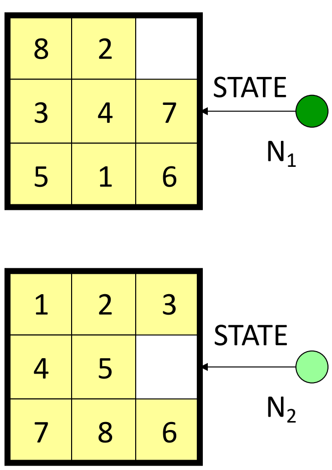
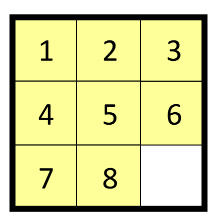
Thanks to Kris Hauser for many slides

Example

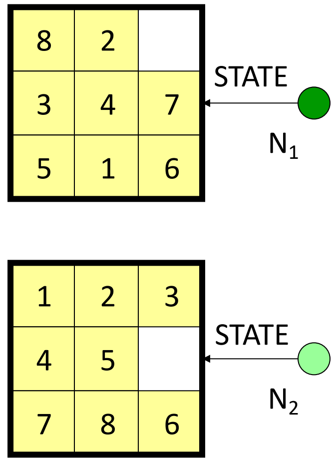
For an uninformed strategy, N_1 and N_2 are just two nodes (at some position in the search tree)





Goal state

Example



For a heuristic strategy counting the number of misplaced tiles, N_2 is more promising than N_1

1	2	3
4	5	6
7	8	

Goal state

4

Heuristic Function

- The heuristic function $h(N) \geq 0$ estimates the cost to go from STATE(N) to a goal state

  Value is ***independent*** of the current search tree; it depends only on STATE(N) and the goal test GOAL

- Example:

5		8
4	2	1
7	3	6

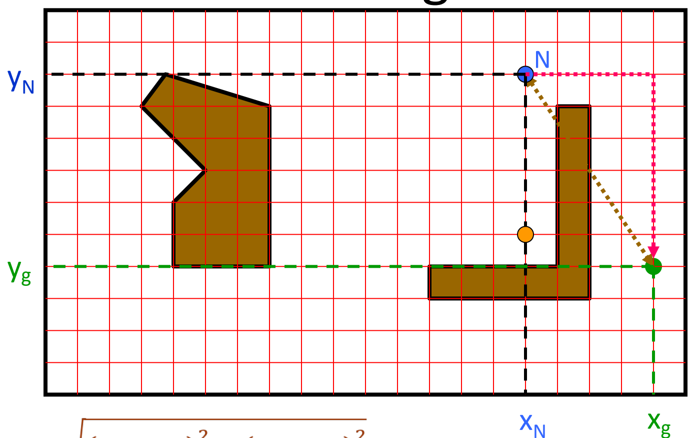
STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- h(N) = number of misplaced numbered tiles = 6
- [Why is it an estimate of the distance to the goal?]

Robot Navigation



$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$ (L_2 or Euclidean distance)

$$h_2(N) = |x_N - x_g| + |y_N - y_g|$$ (L_1 or Manhattan distance)

Informed/Heuristic Search

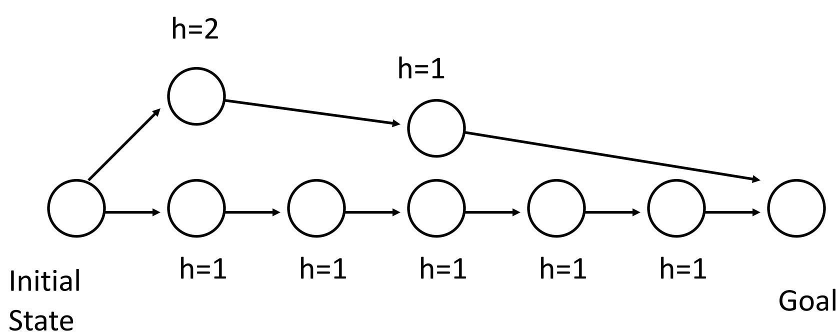
- Idea: Give the search algorithm hints
- Heuristic function: h(x)
- h(x) = estimate of cost to goal from x
- If h(x) is 100% accurate, then we can find the goal in O(bd) time

- How do we use this?

Greedy Best First Search

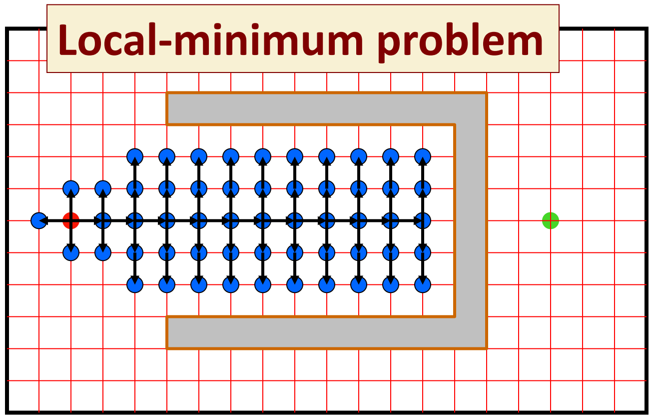
- Expand node with lowest h(x)
- (Implement priority queue on h)
- Optimal if h(x) is 100% correct
- How can we get into trouble with this?

What Price Greed?



What's broken with greedy search?

Best-First ≠ Efficiency



f(N) = h(N) = straight distance to the goal

A*

- Path cost so far: g(x)
- Total cost estimate: f(x) = g(x) + h(x)
- Maintain frontier as a ***priority queue*** (on f)
- O(bd) time if h is 100% accurate
- We want h to be an *admissible* heuristic
- Admissible: never overestimates cost
- Why admissible?

    (guarantees optimality, completeness of A*!)

8-Puzzle Heuristics

5		8
4	2	1
7	3	6

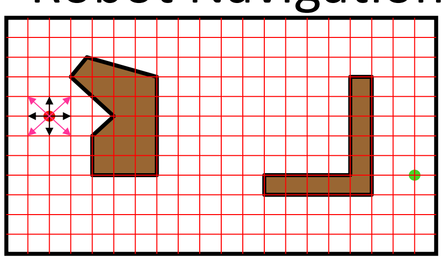
STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles = 6 is ???

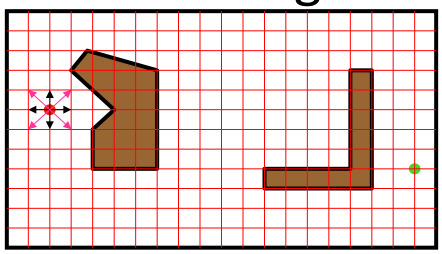
Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$

Robot Navigation Heuristics

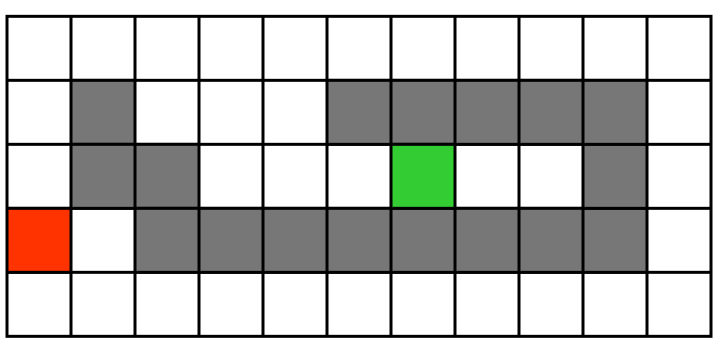


Cost of one horizontal/vertical step = 1

Cost of one diagonal step = $\sqrt{2}$

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is ???

Robot Navigation



Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(greedy, not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(greedy, not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

Robot Navigation

f(N) = g(N)+h(N), with h(N) = Manhattan distance to goal
(A*)

8+3	7+4	6+3	5+6	4+7	3+8	2+9	3+10	4	5	6
7+2		5+6	4+7	3+8						5
6+1			3	2+9	1+10	0+11	1	2		4
7+0	6+1									5
8+1	7+2	6+3	5+4	4+5	3+6	2+7	3+8	4	5	6

Some A* Properties

- Admissibility implies h(x)=0 if x is a goal state
- Above implies f(x)=cost to goal if x is a goal state and x is popped off the queue
- What if h(x)=0 for all x?
  - Is this admissible?
  - What does the algorithm do?

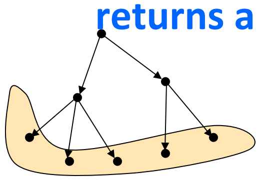
Result #1

A* is complete and optimal

[This result holds if nodes revisiting states are not discarded – otherwise you might find a shortcut and then discard it.]

Proof (1/2)

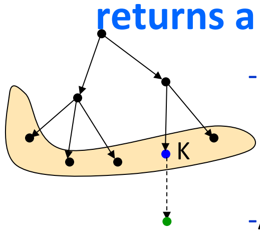
- **If a solution exists, A* terminates and returns a solution**



- For each node N on the frontier, $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\epsilon$, where d(N) is the depth of N in the tree

Proof (1/2)

- **If a solution exists, A* terminates and returns a solution**



- For each node N on the frontier, $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\epsilon$, where $d(N)$ is the depth of N in the tree

- As long as A* hasn't terminated, a node K on the frontier lies on a solution path

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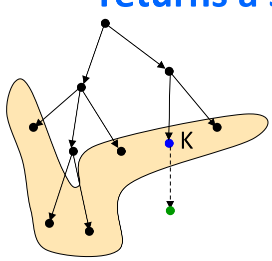


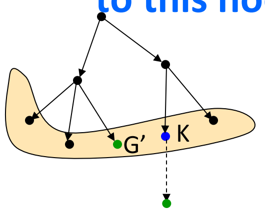
- For each node N on the frontier, $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\epsilon$, where $d(N)$ is the depth of N in the tree

- As long as A* hasn't terminated, a node K on the frontier lies on a solution path

- Since each node expansion increases the length of one path, K will eventually be selected for expansion, unless a solution is found along another path

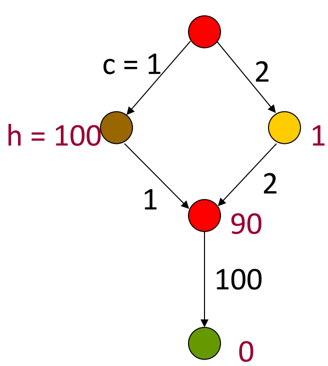
Proof (2/2)

- **Whenever A* pops a goal node, the path to this node is optimal**



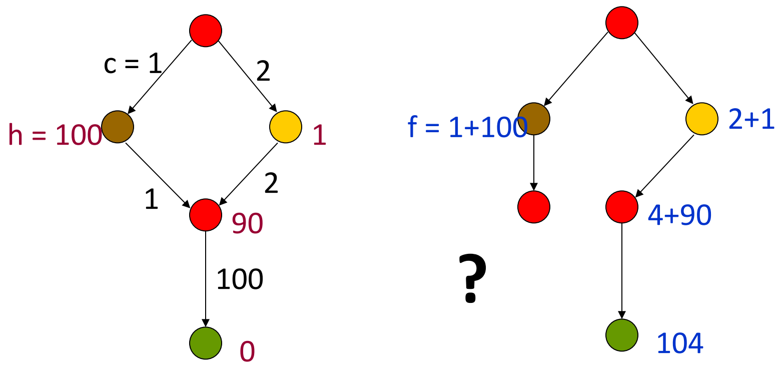
- C*= cost of the optimal solution path
- G’: non-optimal goal node in the frontier
  f(G’) = g(G’) + h(G’) = c(G’) > C*
- A node K in the frontier lies on an optimal path:
  f(K) = g(K) + h(K) ≤ C*
- So, G’ will not be selected for expansion

What to do with revisited states?



The heuristic h is clearly admissible

What to do with revisited states?

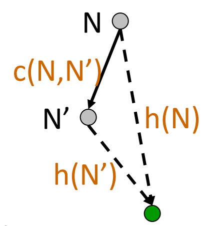


If we discard this new node, then the search algorithm expands the goal node next and returns a non-optimal solution

- Not harmful to discard a node revisiting a state
  if cost of the new path state is $\geq$ cost of previous path
  [so, in particular, one can discard a node if it re-visits a state already visited by one of its ancestors – compare w/DFS]
- If A* pushes revisited states, it remains optimal, but states may be re-visited multiple times
  [the size of the search tree can be exponential in number of visited states]
- Fortunately, for a large family of admissible heuristics – consistent heuristics – there is a much more efficient way to handle revisited states

Consistent Heuristic

- An admissible heuristic h is consistent (or monotone) if for each node N and each child N' of N: $h(N) \leq c(N,N') + h(N')$

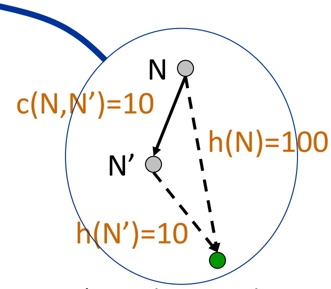


(triangle inequality)

→ Intuition: a consistent heuristics becomes more precise as we get deeper in the search tree

Consistency Violation

If h tells us that N is 100 units from the goal, then moving from N along an arc costing 10 units should **not** lead to a node N' that h estimates to be 10 units away from the goal

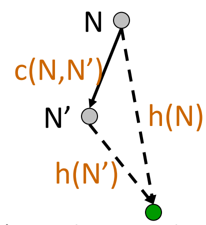


(triangle inequality violation)

Consistent Heuristic (alternative definition)

- A heuristic h is consistent (or monotone) if

1. for each node N and each child N' of N:
   $h(N) \leq c(N,N') + h(N')$
2. for each goal node G:
   $h(G) = 0$



(triangle inequality)

Admissibility and Consistency

- Any consistent heuristic is also admissible
- An admissible heuristic may not be consistent, but many admissible heuristics are

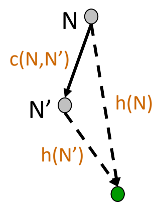
8-Puzzle

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

goal



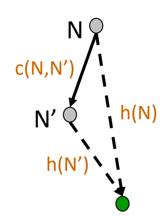
$h(N) \leq c(N,N') + h(N')$

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position

are both consistent (why?)

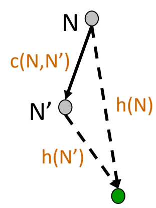
Reasoning About Consistency

- Example: Manhattan Distance in 8-puzzle
  - MD(N,G) <= MD(N,N')+MD(N',G)
  - h(N) = MD(N,G)
  - h(N') = MD(N',G)
  - h(N) <= MD(N,N')+h(N')
  - C(N,N') >= MD(N,N')
  - h(N) <= C(N,N')+h(N')
- Note: Not just showing that h obeys triangle inequality between pairs of states

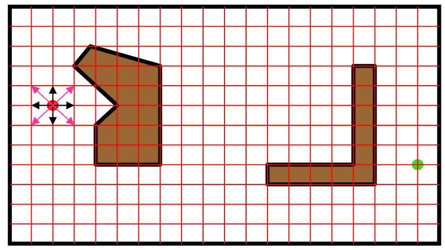


$h(N) \leq c(N,N') + h(N')$

Robot Navigation



h(N) ≤ c(N,N') + h(N')



Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

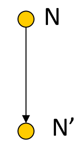
$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$ is consistent

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is consistent if moving along diagonals is not allowed, and not consistent otherwise

Result #2

- If h is consistent, then whenever A* expands a node, it has already found an optimal path to this node's state

Proof (1/2)

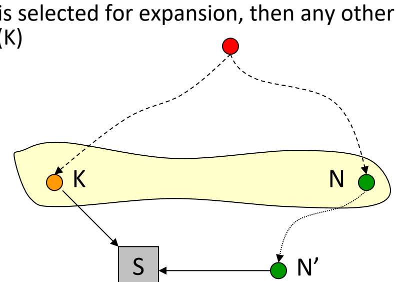


1. Consider a node N and its child N'
   Since h is consistent: $h(N) \le c(N,N')+h(N')$

   $f(N) = g(N)+h(N) \le g(N)+c(N,N')+h(N') = f(N')$
   So, f is non-decreasing along any path

Proof (2/2)

2. If a node K is selected for expansion, then any other node N in the frontier has $f(N) \ge f(K)$



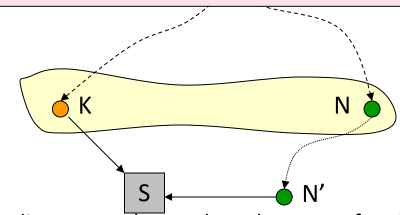
- If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

  $f(N') \ge f(N) \ge f(K)$ and $h(N') = h(K)$

  So, $g(N') \ge g(K)$

Result #2

If h is consistent, then whenever A* expands a node, it has already found an optimal path to this node's state

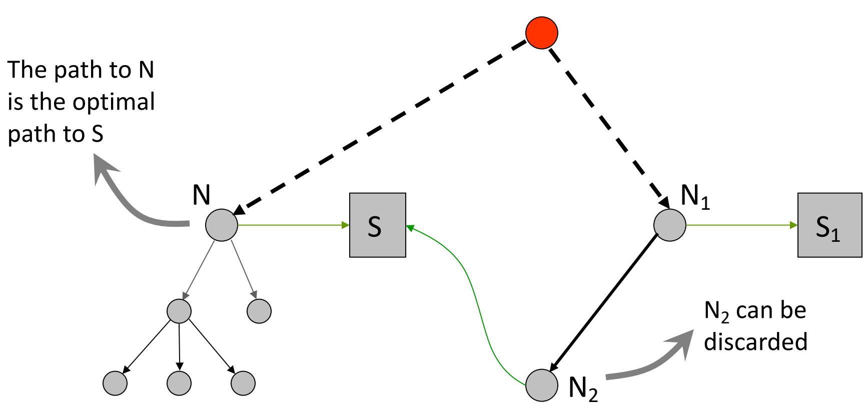


- If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

  $f(N') \geq f(N) \geq f(K)$ and $h(N') = h(K)$

  So, $g(N') \geq g(K)$

Implication of Result #2



Revisited States with Consistent Heuristic (Modified Search Algorithm #3)

- When a node is expanded, store its state into VISITED
- When a new node N' is generated:
  - If STATE(N') is in VISITED, discard N'
  - If there exists a node N'' in the frontier such that STATE(N'') = STATE(N'), discard the node – N' or N'' – with the largest f (or, equivalently, g)

Not as important – can safely ignore these checks and just push onto the queue – Why?

Heuristic Accuracy

- Let h_1 and h_2 be two consistent heuristics such that for all nodes N:

$$h_1(N) \leq h_2(N)$$

- h_2 is said to be more accurate (or more informed) than h_1

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of distances of every tile to its goal position
- h_2 is more accurate than h_1

Result #3

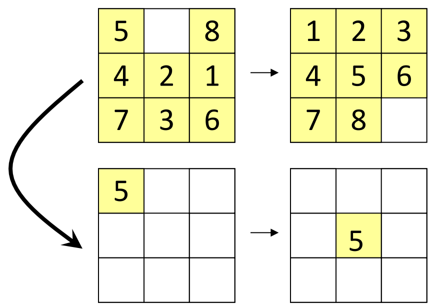
- Let h_2 be more accurate than h_1
- Let A_1* be A* using h_1
  and A_2* be A* using h_2
- Whenever a solution exists, all the nodes expanded by A_2*, except possibly for some nodes such that
  $f_1(N) = f_2(N) = C^*$ (cost of optimal solution)
  are also expanded by A_1*

Proof

- C* = cost of optimal solution
- Every node N such that $f(N) < C^*$ is eventually expanded. No node N such that $f(N) > C^*$ is ever expanded
- Every node N such that $h(N) < C^*-g(N)$ is eventually expanded. So, every node N such that $h_2(N) < C^*-g(N)$ is expanded by A_2*. Since $h_1(N) \leq h_2(N)$, N is also expanded by A_1*
- If there are several nodes N such that $f_1(N) = f_2(N) = C^*$ (such nodes include the optimal goal nodes, if there exists a solution), A_1* and A_2* may or may not expand them in the same order (until one goal node is expanded)

How to create good heuristics?

- By solving relaxed problems at each node
- In the 8-puzzle, the sum of the distances of each tile to its goal position (h_2) corresponds to solving 8 simple problems:



d_i is the length of the shortest path to move tile i to its goal position, ignoring the other tiles, e.g., $d_5 = 2$

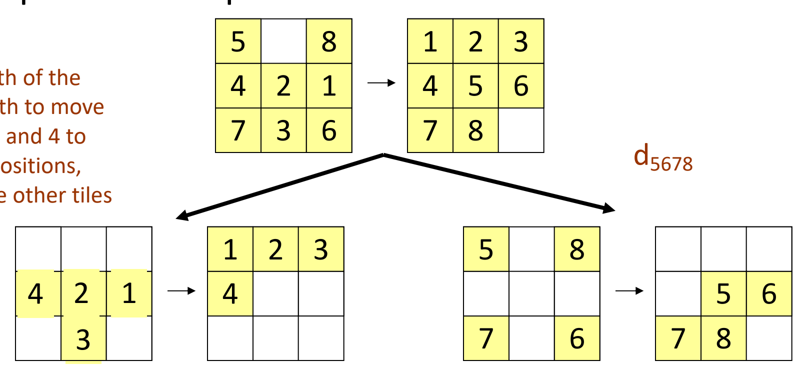
$$h_2(N) = \sum_{i=1}^{8} d_i(N)$$

- It ignores negative interactions among tiles

Can we do better?

- For example, we could consider two more complex relaxed problems:

d_{1234} = length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles

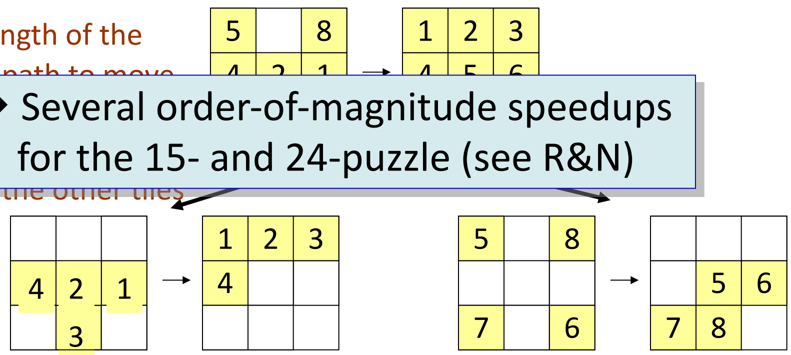


- → $h = d_{1234} + d_{5678}$ [disjoint pattern heuristic]
- How to compute d_{1234} and d_{5678}?

Can we do better?

- For example, we could consider two more complex relaxed problems:

d_{1234} = length of the shortest path to move tiles 1… their … ignoring the other tiles



→ Several order-of-magnitude speedups for the 15- and 24-puzzle (see R&N)

- → $h = d_{1234} + d_{5678}$ [disjoint pattern heuristic]
- These distances are pre-computed and stored

[Each requires generating a tree of 3,024 nodes/states (breadth-first search)]

Effective Branching Factor

- Used as measure the effectiveness of h
- Let n be the total number of nodes expanded by A* for a particular problem and d the depth of the solution
- The effective branching factor b^* is defined by fitting: $n = 1 + b^* + (b^*)^2 + ... + (b^*)^d$

Experimental Results

(see R&N for details)

- 8-puzzle with:
  - h_1 = number of misplaced tiles
  - h_2 = sum of distances of tiles to their goal positions
- Random generation of many problem instances
- Average effective branching factors (number of expanded nodes):

d	IDDFS	A_1*	A_2*
2	2.45	1.79	1.79
6	2.73	1.34	1.30
12	2.78 (3,644,035)	1.42 (227)	1.24 (73)
16	--	1.45	1.25
20	--	1.47	1.27
24	--	1.48 (39,135)	1.26 (1,641)

Memory-bounded Search: Why?

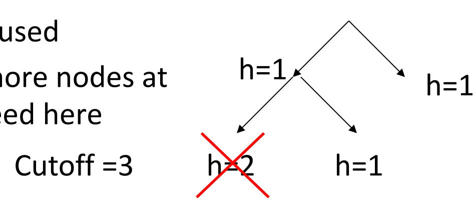
- We run out of memory before we run out of time
- Problem: Need to remember entire search horizon
- Solution: Remember only a partial search horizon
- Issue: Maintaining optimality, completeness
- Issue: How to minimize time penalty
- Details: Not emphasized in class, but worth a skim so that you are aware of the issues

Iterative Deepening A* (IDA*)

- Idea: Reduce memory requirement of A* by applying cutoff on values of f
- Consistent heuristic function h
- Algorithm IDA*:
  - Initialize cutoff to f(initial-node)
  - Repeat:
    - Perform cost-limited search by expanding all nodes N such that f(N) ≤ cutoff
    - Reset cutoff to smallest value f of non-expanded (leaf) nodes

Advantages/Drawbacks of IDA*

- Advantages:
  - Still complete and optimal
  - Requires less memory than A*
  - Avoids the overhead to sort the frontier (priority queue)
- Drawbacks:
  - Discards a lot of information when it restarts
  - Available memory is poorly used
  - IDDFS expands factor of b more nodes at each iteration; not guaranteed here

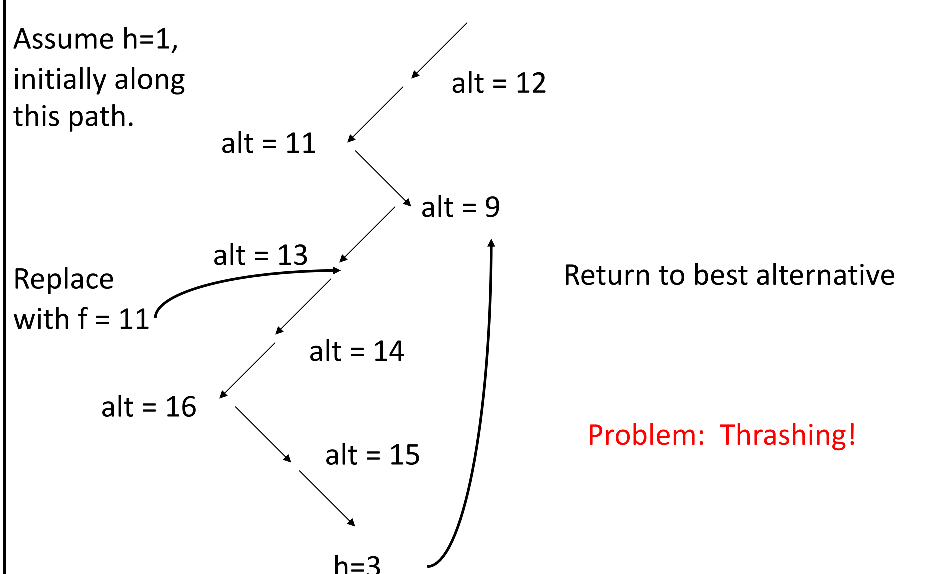


RBFS

- Recursive best first search
- Objective: Linear space without discarding as much information as IDA*

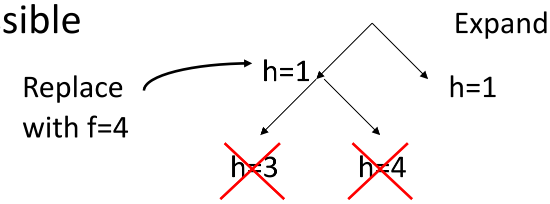
- Idea: Remember best alternative
- Rewind, try alternatives if “best first” path gets too expensive
- Remember costs on the way back up

RBFS



SMA*

- Idea: Use all of available memory
- Discard the *worst* leaf when memory starts to run out, to make room for new leaves
- Values get backed up to parents
- Optimal if solution fits in memory
- Complete
- Thrashing still possible



Painful to implement ☹

Recap

- Heuristics change how we think about search
- A* is optimal, complete
- Dramatic improvements in efficiency possible with good heuristics
- Many extensions possible, e.g., dealing with limited memory