What is Search?

- Search is a basic problem-solving method
  - We start in an initial state
  - We examine states that are (usually) connected by a sequence of actions to the initial state

- Note: Search is (usually) a thought experiment (separate topic: Real Time Search)

- We aim to find a solution, which is a sequence of actions that brings us from the initial state to the goal state, possibly minimizing cost
Search vs. Web Search

• When we issue a search query using Google, does Google really go poking around the web for us?

• Not in real time!
• Google spiders the web continually, caches results
• Uses page rank algorithm to find the most “popular” web pages that are consistent with your query

Overview

• Problem Formulation

• Uninformed Search – constant cost
  – DFS, BFS, IDDFS, etc.

• Non-constant cost
Problem Formulation

• Components of a search problem
  – State space & initial state
  – Actions
  – Goal Test
  – Edge costs (constant or varying per edge?)
• Optimal solution = lowest path cost to goal

Example: Path Planning, e.g. Google Maps

Find shortest source to destination using available roads
Other Search Problems

- Drug design
- Logistics
  - Route planning
  - Tour Planning
- Assembly sequencing
- Internet routing
- Robot motion/path planning

Robot Path Planning

What is the state space?
Formulation #1

Cost of one horizontal/vertical step = 1
Cost of one diagonal step = \sqrt{2}

Optimal Solution

This path is the shortest in the discretized state space, but not in the original continuous space.
Formulation #2

Cost of one step: length of segment

Visibility graph

Formulation #2

Cost of one step: length of segment
The shortest path in this state space is also the shortest in the original continuous space.

**Take Home Points**

- **States** = modeling choice about the world

- **Trade offs** often exist:
  - Example 1: Discretization is easy to work with, but optimal solution to may be suboptimal in the real world
  - Example 2: More clever representations may require ingenuity to discover, or use, but may have benefits in real world

- **Always keep** modeling and solving **distinct in your head**
Basic Search Concepts

- **Search tree**: Internal representation of our progress
- **Nodes**: Places in search tree (states exist in the problem space)
- **Actions**: Connect states to next states (nodes to nodes)
- **Expansion**: Generation of next states (nodes)
- **Arc cost**: Cost of moving from one state to another
- **Frontier**: Set of nodes visited, but not expanded
- **Branching factor**: Max no. of successors = b
- **Goal depth**: Depth of shallowest goal = d (root is depth 0, possibility of multiple goal states!)

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Example: 8-Puzzle

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

Initial state

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Goal state

**State**: Arrangement of 8 numbered tiles & empty tile on a 3x3 board
15-Puzzle

- Introduced (?) in 1878 by Sam Loyd, who dubbed himself “America’s greatest puzzle-expert”

15-Puzzle

- Sam Loyd offered $1,000 of his own money to the first person who would solve the following problem:
How big is the state space of the \((n^2-1)\)-puzzle?

- 8-puzzle \(\rightarrow 9! = 362,880\) states
- 15-puzzle \(\rightarrow 16! \sim 2.09 \times 10^{13}\) states
- 24-puzzle \(\rightarrow 25! \sim 10^{25}\) states

- But only half of these states are reachable from any given state (but you may not know that in advance)

- No one ever won the prize !!
Searching the State Space

• Often infeasible (or too expensive) to build complete representation of the state graph

• Key difference from algorithms class, where it is typically assumed that graph fits in memory

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8-, 15-, 24-Puzzles

- **8-puzzle** → 362,880 states
- **15-puzzle** → $2.09 \times 10^{13}$ states
  - ~ 55 hours
- **24-puzzle** → $10^{25}$ states
  - > $10^9$ years

100 million states/sec
Intractability

• Constructing the full state graph is intractable for many interesting problems
• n-puzzle: (n+1)! states

Tractability of search hinges on the ability to explore only a tiny portion of the state graph!

Searching the State Space
Searching the State Space

Search tree
Searching the State Space

Search tree
Searching the State Space

If states are allowed to be revisited, the search tree may be infinite even when the state space is finite.
Data Structure of a Node

- **STATE**
- **PARENT-NODE**
- **CHILDREN**
- **BOOKKEEPING**

**Depth of a node N**

\[ \text{Depth of a node } N = \text{length of path from root to } N \]

\[ (\text{depth of the root } = 0) \]

Node expansion

- The **expansion** of a node N of the search tree consists of:
  - Evaluating the successor function on \( \text{STATE}(N) \)
  - Generating a child of N for each state returned by the function

- **node generation ≠ node expansion**
Frontier of Search Tree

- The **frontier** is the set of all search nodes that haven’t been expanded yet

Search Strategy

- The **frontier** is the set of all search nodes that haven’t been expanded yet
- Implemented as a priority queue FRONTIER
  - INSERT(node, FRONTIER)
  - REMOVE(FRONTIER)
- The ordering of the nodes in FRONTIER defines the search strategy
Generic Tree Search

**TREE-SEARCH(initial-state)**
1. If GOAL?(initial-state) then return initial-state
2. INSERT(initial-node, FRONTIER)
3. Repeat:
4. If empty(FRONTIER) then return failure
5. \( N \leftarrow \text{REMOVE}(\text{FRONTIER}) \)
6. \( s \leftarrow \text{STATE}(N) \)
7. For every state \( s' \) in SUCCESSORS(s)
8. Create a new node \( N' \) as a child of \( N \)
9. If GOAL?(s') then return path or goal state
10. INSERT(N', FRONTIER)

Expansion of \( N \)

Solution to the Search Problem

- A **solution** is a path connecting the initial node to a goal node (any goal)
- The **cost** of a path is the sum of the arc costs along this path
- An **optimal** solution is a solution path of minimum cost
- There might be no solution!

Nota bene: Typically assume costs > 0
Algorithm Performance Measures

• Completeness:
  – Does it find a solution when one exists?

• Optimality:
  – Does it return a min cost path whenever solution exists?

• Complexity (space or time):
  – Resources required by the algorithm

Breadth-First Search

• FRONTIER is a FIFO Queue

FRONTIER: 1, 2, 3, 4, 5, 6, 7
FRONTIER = (1)
Breadth-First Search

- FRONTIER is a FIFO Queue

FRONTIER = (2, 3)
Breadth-First Search

- FRONTIER is a FIFO Queue

FRONTIER = (4, 5, 6, 7)

BFS Properties

- Completeness: Y
- Optimality: (Y for constant cost, N for arbitrary cost)
- Time complexity: $O(b^{d+1})$
- Space complexity: $O(b^d)$

Note: We are counting nodes generated in time complexity; textbook may be counting nodes expanded.
How bad is exponential in d?

<table>
<thead>
<tr>
<th>d</th>
<th># Nodes</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>111</td>
<td>.01 msec</td>
<td>11 Kbytes</td>
</tr>
<tr>
<td>4</td>
<td>11,111</td>
<td>1 msec</td>
<td>1 Mbyte</td>
</tr>
<tr>
<td>6</td>
<td>~10^6</td>
<td>1 sec</td>
<td>100 Mb</td>
</tr>
<tr>
<td>8</td>
<td>~10^8</td>
<td>100 sec</td>
<td>10 Gbytes</td>
</tr>
<tr>
<td>10</td>
<td>~10^10</td>
<td>2.8 hours</td>
<td>1 Tbyte</td>
</tr>
<tr>
<td>12</td>
<td>~10^12</td>
<td>11.6 days</td>
<td>100 Tbytes</td>
</tr>
<tr>
<td>14</td>
<td>~10^14</td>
<td>3.2 years</td>
<td>10,000 Tbytes</td>
</tr>
</tbody>
</table>

Assumptions: b = 10; 1,000,000 nodes/sec; 100 bytes/node

Bi-directional Search

\[ b^{d/2} + b^{d/2} \ll b^d \]
Issues with Bi-directional Search

- Uniqueness of goal
  - Suppose goal is parking your car
  - Huge no. of possible goal states (configurations of other vehicles)

- Invertability of actions

Depth-First Search

- FRONTIER is a LIFO Queue

![Diagram of Depth-First Search]
Depth-First Search

- FRONTIER is a LIFO Queue

1

2

3

4

5

FRONTIER = (2, 3)

FRONTIER = (4, 5, 3)
Depth-First Search

• FRONTIER is a LIFO Queue
Depth-First Search

- FRONTIER is a LIFO Queue
Depth-First Search

• FRONTER is a LIFO Queue

1
2
4
5

3

Depth-First Search

• FRONTER is a LIFO Queue

1
2
4
5

3

Depth-First Search

- FRONTIER is a LIFO Queue
DFS Properties

- Completeness: (Y for finite trees, N for infinite trees)
- Optimality: N
- Time complexity: $O(b^{m+1})$ (m = depth we hit, m>d?)
- Space complexity: $O(bm)$ (bounded for trees)

Iterative Deepening

- Want:
  - DFS memory requirements
  - BFS optimality, completeness
- Idea:
  - Do a depth-limited DFS for depth m
  - Iterate over m
Iterative Deepening

Note: The IDDFS slides are animated, showing DFS running down to the red line on each slide.
Iterative Deepening

IDDFS Properties

- Completeness: \( \exists \) (whenever BFS is optimal)
- Optimality: \( O(b^{d+2}) \)
- Time complexity: \( O(b^{d+2}) \)
- Space complexity: \( O(bd) \)
IDDFS vs. BFS

Theorem: IDDFS expands no more than twice as many nodes for a binary tree as BFS.

Proof: Assume the tree bottoms out at depth d, BFS generates:

$2^{d+1} - 1$

In the worst case, IDDFS does no more than:

$\sum_{i=0}^{d} (2^{i+1} - 1) = \sum_{i=0}^{d} 2^{i+1} - \sum_{i=0}^{d} 1 = 2(2^{d+1} - 1) - (d + 1) < 2(2^{d+1} - 1) = 2 \times BFS(d)$

What about b-ary trees?      IDDFS relative cost is lower!

Non-constant Costs

• Arcs between states can have variable costs

• The cost of the path to each node $N$ is $g(N) = \Sigma$ costs of arcs

• Breadth-first is no longer optimal
Uniform-Cost Search (UCS)

- Expand node in FRONTIER with the cheapest path so far, i.e., frontier is a priority queue prioritized on $g(N)$

Search Algorithm #2

$\text{TREE-SEARCH2}(\text{initial-state})$

1. If $\text{GOAL?}(\text{initial-state})$ then return $\text{initial-state}$
2. $\text{INSERT}(\text{initial-node,FRONTIER})$
3. Repeat:
   4. If empty(FRONTIER) then return $\text{failure}$
   5. $N \leftarrow \text{REMOVE}(\text{FRONTIER})$
   6. $s \leftarrow \text{STATE}(N)$
   7. If $\text{GOAL?}(s)$ then return path or goal state
   8. For every state $s'$ in SUCCESSORS( $s$ )
      9. Create a new node $N'$ as a child of $N$
      10. $\text{INSERT}(N',\text{FRONTIER})$

The goal test is applied to a node when this node is expanded, not when it is generated.

Now, UCS is optimal!
Avoiding Revisited States
(Searching Graphs)

- Requires comparing state descriptions
- Breadth-first search:
  - Store all states associated with *generated* nodes in VISITED
  - If the state of a new node is in VISITED, then discard the node

Implemented as hash-table (e.g. python dictionary) or as explicit data structure with flags
Explicit Data Structures

- Robot navigation
- VISITED: array initialized to 0, matching grid
- When grid position \((x,y)\) is visited, mark corresponding position in VISITED as 1
- **Size of the entire state space!**

Avoiding Revisited States in DFS

- Depth-first search:
  - Solution 1:
    - Store all states in current path in VISITED
    - If the state of a new node is in VISITED, then discard the node
  - Only avoids loops

  - Solution 2:
    - Store all generated states in VISITED
    - If the state of a new node is in VISITED, then discard the node
  - Same space complexity as breadth-first!
Avoiding Revisited States in Uniform-Cost Search

• For any state S, when the first node N such that \( \text{STATE}(N) = S \) is expanded, the path to N is the best path from the initial state to S

• So:
  – When a node is **expanded**, store its state into VISITED
  – When a new node N is generated:
    • If \( \text{STATE}(N) \) is in VISITED, discard N
    • If there exists a node \( N' \) in the frontier such that \( \text{STATE}(N') = \text{STATE}(N) \), discard the node -- N or \( N' \) -- w/highest cost

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Search Algorithm #3

**GRAPH-SEARCH**(initial-state)

1. If \( \text{GOAL?}(\text{initial-state}) \) then return initial-state
2. INSERT(initial-node,FRONTIER)
3. Repeat:
4. If empty(FRONTIER) then return failure
5. \( N \leftarrow \text{REMOVE}(\text{FRONTIER}) \)
6. \( s \leftarrow \text{STATE}(N) \)
7. Add s to VISITED
8. If \( \text{GOAL?}(s) \) then return path or goal state
9. For every state \( s' \) in SUCCESSORS( )
10. Create a new node \( N' \) as a child of N
11. If \( s' \) is in VISITED then discard \( N' \)
12. If there is \( N'' \) in FRONTIER with \( \text{STATE}(N') = \text{STATE}(N'') \)
13. If \( g(N'') \) is lower than \( g(N') \) then discard \( N' \)
14. Else discard \( N'' \)
15. INSERT(\( N', \text{FRONTIER} \))
Uninformed Search Summary

• Many variations on same basic algorithm

• Key differences:
  – How frontier is implemented (FIFO, LIFO, priority queue)
  – When goal test is applied
  – Whether visited list is maintained

• Big impact on:
  – Completeness
  – Optimality
  – Complexity