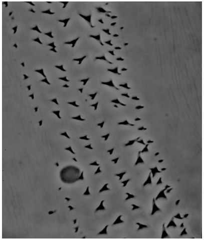
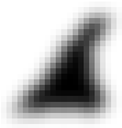
Correlation, Convolution, Filtering

COMPSCI 527 — Computer Vision

Outline

Template Matching





Normalized Cross-Correlation

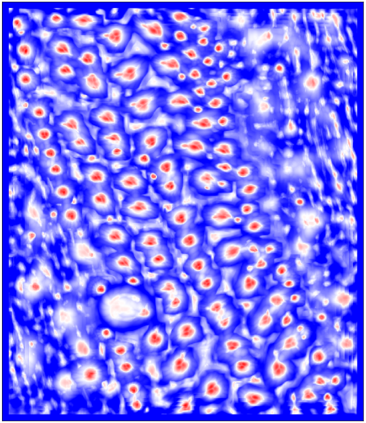
$$\rho(r,c) = \boldsymbol{\tau}^T \boldsymbol{\omega}(r,c)$$

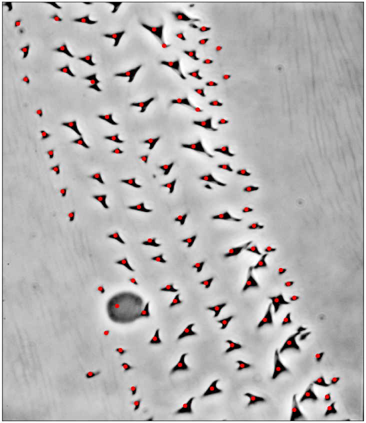
$$\boldsymbol{\tau} = \frac{\mathbf{t} - m_{\mathbf{t}}}{\|\mathbf{t} - m_{\mathbf{t}}\|} \quad \text{and} \quad \boldsymbol{\omega}(r,c) = \frac{\mathbf{w}(r,c) - m_{\mathbf{w}(r,c)}}{\|\mathbf{w}(r,c) - m_{\mathbf{w}(r,c)}\|}$$

$$-1 \leq \rho(r,c) \leq 1$$

$$\begin{aligned} \rho = 1 \quad &\Leftrightarrow \quad W(r,c) = \alpha T + \beta\,, \quad \alpha > 0 \\ \rho = -1 \quad &\Leftrightarrow \quad W(r,c) = \alpha T + \beta\,, \quad \alpha < 0 \end{aligned}$$

Results





Cross-Correlation

(ignoring normalization for simplicity)

$$J(r, c) = \mathbf{t}^T \mathbf{w}(r, c)$$

Code, Math

```
for r = 1:m
  for c = 1:n
    J(r, c) = 0
    for u = -h:h
      for v = -h:h
        J(r, c) = J(r, c) + T(u, v) * I(r+u, c+v)
      end
    end
  end
end
```

$$J(r,c) = \sum_{u=-h}^{h} \sum_{v=-h}^{h} I(r+u, c+v) T(u,v)$$

Convolution

Correlation:

$$J(r,c) = \sum_{u=-h}^{h} \sum_{v=-h}^{h} I(r+u, c+v)T(u,v)$$

Convolution:

$$J(r,c) = \sum_{u=-h}^{h} \sum_{v=-h}^{h} I(r-u, c-v)H(u,v)$$

Same as

$$J(r,c) = \sum_{u=-h}^{h} \sum_{v=-h}^{h} I(r+u, c+v)H(-u,-v)$$

Convolution with *kernel* $H(u,v)$ is correlation with *template* $T(u,v) = H(-u,-v)$

What's the Big Deal?

$$\text{Simplify} \quad J(r,c) = \sum_{u=-h}^{h} \sum_{v=-h}^{h} I(r-u, c-v) H(u,v)$$

$$\text{to} \quad J(r,c) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} I(r-u, c-v) H(u,v)$$

Changes of variables $u \leftarrow r - u$ and $v \leftarrow c - v$

$$J(r,c) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} H(r-u, c-v) I(u,v)$$

Convolution commutes: $I * H = H * I$

(Correlation does not)

Importance of Convolution in Mathematics

- Polynomials: coefficients of product are “full” convolutions of coefficients:

  $$P(x) = p_0 + p_1 x + \ldots + p_m x^m$$
  $$Q(x) = q_0 + q_1 x + \ldots + q_n x^n$$
  $$R(x) = p_0 q_0 + (p_0 q_1 + p_1 q_0) x + \ldots + p_m q_n x^{m+n}$$

- Example:

  $$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 \quad \rightarrow \quad (p_0, p_1, p_2, p_3)$$
  $$Q(x) = q_0 + q_1 x + q_2 x^2 \quad \rightarrow \quad (q_0, q_1, q_2)$$

  Convolve (p_0, p_1, p_2, p_3) with (q_0, q_1, q_2) to get $(r_0, r_1, r_2, r_3, r_4, r_5)$

Important Consequence

- Discrete Fourier transform is a polynomial:
  $p = (p_0, \ldots, p_{n-1})$
- $\mathcal{F}[p](\ell) = p_0 + p_1 z + \ldots + p_{n-1} z^{n-1}$ where $z = \frac{1}{n} e^{-i2\pi\ell/n}$
- All of spectral signal theory follows
- Example: The Fourier transform of a convolution is the product of the Fourier transforms
- [We will not see this]

Image Boundaries: “Valid” Convolution

- Full overlap of image and kernel
- If I is $m \times n$ and H is $k \times \ell$, then J is $(m - k + 1) \times (n - \ell + 1)$

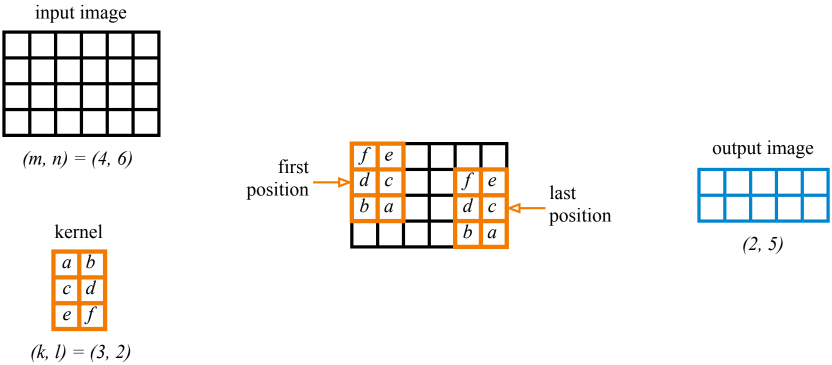


Image Boundaries: “Full” Convolution

- Any non-empty overlap of image and kernel
- If I is $m \times n$ and H is $k \times \ell$, then J is $(m+k-1) \times (n+\ell-1)$
  [Pad with either zeros or copies of boundary pixels]

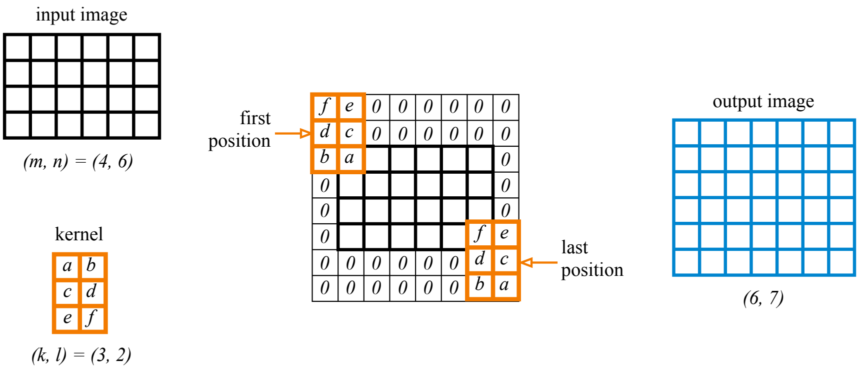
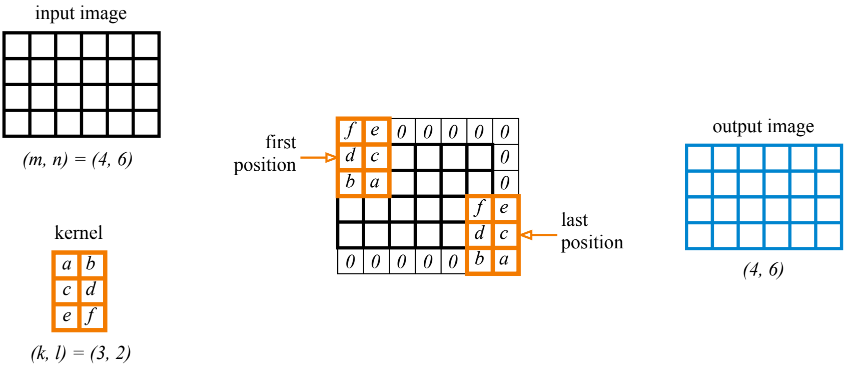


Image Boundaries: "Same" Convolution

- Require the output to have the same size as the input
- If I is $m \times n$ and H is $k \times \ell$, then J is $m \times n$

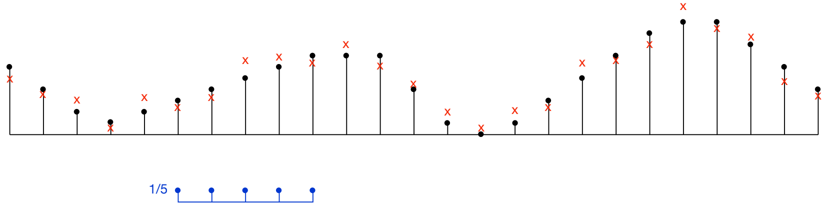


Filters

- What is convolution for?
  - Smoothing for noise reduction
  - Image differentiation
  - Convolutional Neural Networks (CNNs)
  - ...
- Smoothing and differentiation are examples of *filtering*: Local, linear image $\rightarrow$ image transformations

Smoothing for Noise Reduction

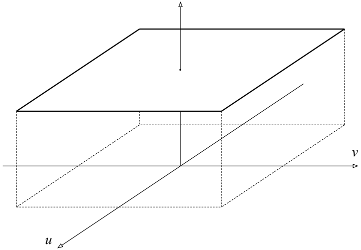
- Assume: Image varies slowly enough to be *locally linear*
- Assume: Noise is zero-mean and white



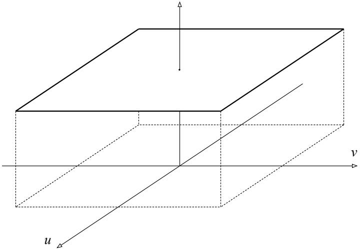
Averaging as Convolution

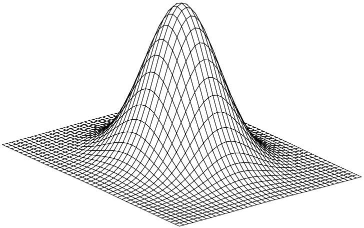
$J(c) = \frac{1}{2h+1} \sum_{v=-h}^{h} I(c-h)$ is the same as
$J(c) = \sum_{v=-h}^{h} I(c-h)H(c)$ where $H(c) = \frac{1}{2h+1}[1, \ldots, 1]$,
a convolution with the *box kernel*

Box kernel in two dimensions:



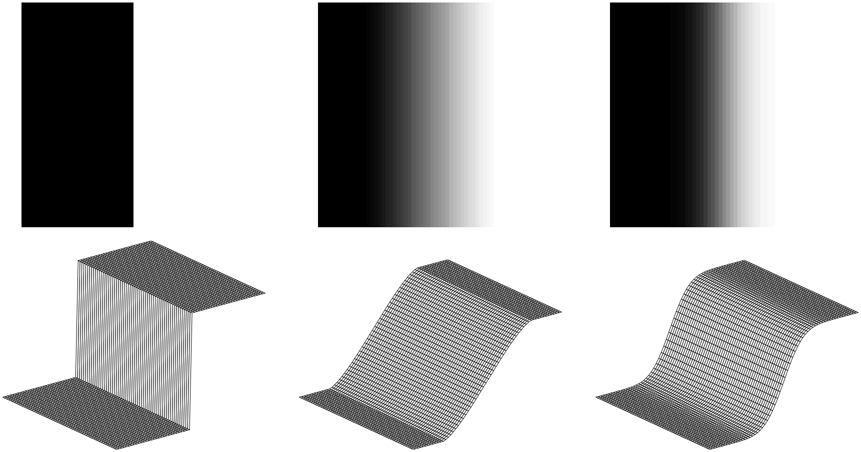
Box versus Gaussian Kernel





- The Gaussian kernel does a *weighted* average
- Emphasizes nearby values more than distant ones
- Blurs less than the box kernel for the same averaging effect

Box versus Gaussian Kernel



Truncation

$G(u, v) = e^{-\frac{1}{2}\frac{u^2+v^2}{\sigma^2}}$

- The larger σ, the more smoothing
- u, v integer, and cannot keep them all
- Truncate at 3σ or so

$e^{-\frac{3^2}{2}} \approx 0.01$

Normalization

$$G(u, v) = e^{-\frac{1}{2}\frac{u^2+v^2}{\sigma^2}}$$

- We want $I * G \approx I$
- For $I = c$ (constant), $I * G = I$
- Normalize by computing $\gamma = 1 * G$, and then let $G \leftarrow G/\gamma$

Separability

- A kernel that satisfies $H(u, v) = h(u)\ell(v)$ is *separable*
- The Gaussian is separable with $h = \ell$:

$$G(u, v) = e^{-\frac{1}{2}\frac{u^2+v^2}{\sigma^2}} = g(u)\, g(v) \quad \text{with} \quad g(u) = e^{-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2}$$

- A separable kernel leads to efficient convolution:

$$\begin{aligned} J(r, c) &= \sum_{u=-h}^{h} \sum_{v=-k}^{k} H(u, v)\, I(r-u, c-v) \\ &= \sum_{u=-h}^{h} h(u) \sum_{v=-k}^{k} \ell(v)\, I(r-u, c-v) \\ &= \sum_{u=-h}^{h} h(u)\, \phi(r-u, c) \quad \text{where} \quad \phi(r, c) = \sum_{v=-h}^{h} \ell(v) I(r, c-v) \end{aligned}$$

Computational Complexity

General: $J(r, c) = \sum_{u=-h}^{h} \sum_{v=-k}^{k} H(u, v)\, I(r - u, c - v)$

Separable: $J(r, c) = \sum_{u=-h}^{h} h(u)\, \phi(r - u, c)$ where

$\phi(r, c) = \sum_{v=-h}^{h} \ell(v) I(r, c - v)$

Let $m = 2h + 1$ and $n = 2k + 1$

General: About $2mn$ operations per pixel

Separable: About $2m + 2n$ operations per pixel

Example:

When $m = n$ (square kernel), the gain is $2m^2/4m = m/2$

With $m = 20$: About 80 operations per pixel instead of 800