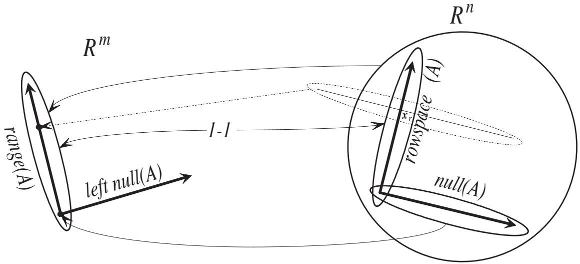
Linear Systems

COMPSCI 527 — Computer Vision

Outline

1. Linear Transformations
2. The Solutions of a Linear System
3. Orthogonal Matrices
4. The Singular Value Decomposition
5. The Pseudoinverse
6. Homogeneous Linear System on the Unit Sphere

The Four Fundamental Spaces of a Matrix



$$A = \begin{bmatrix} \sqrt{3} & \sqrt{3} & 0 \\ -3 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\text{null}(A) = \text{span}([0, 0, 1]^T)$
$\text{row space}(A) = \text{span}(\text{first two rows})$
$\text{range}(A) = \text{span}(\text{first two columns})$
$\text{left null}(A) = \text{span}(\text{cross product of first two columns})$
(comes out to be $\text{span}([-1, 0, \sqrt{3}]^T)$)

$\text{range}(A) \leftrightarrow \text{row space}(A)$

The Solutions of a Linear System

$$A\mathbf{x} = \mathbf{b}$$

where A is $m \times n$, rank r

- Key point:

$$\begin{array}{lcl} \mathbf{b} \notin \text{range}(A) & \Rightarrow & \text{no solutions} \\ \mathbf{b} \in \text{range}(A) & \Rightarrow & \infty^{n-r} \text{ solutions} \end{array}$$

(An *affine space* of solutions)

Compatibility

- Incompatible:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} \qquad (b_3 \neq 0)$$

- Compatible:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$$

Under-Determined System

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Redundant and Invertible Systems

- Redundant:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

- Invertible:

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

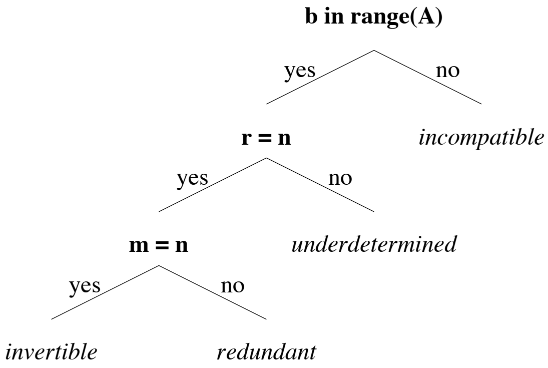
- Inverse:

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3} \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(This is *not* how linear systems are typically solved)

Summary



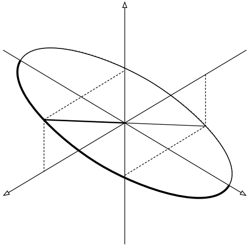
- This is not operational!
- Orthogonal matrices $\rightarrow$ SVD $\rightarrow$ rank, bases for the four spaces
- SVD gives us much more

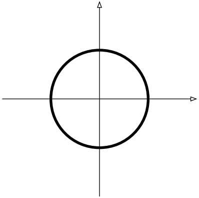
Orthogonal Matrices

- A matrix $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is orthogonal if its *columns* are orthonormal
- Orthonormal: $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$ (orthogonal and unit norm)
- Orthogonal matrices have left-inverse V^T
- *Square* orthogonal matrices have left- and right-inverse V^T
- Orthogonal matrices do not change the norm of vectors: $\|V\mathbf{x}\|^2 = \mathbf{x}^T V^T V \mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2$

The Singular Value Decomposition: Geometry

$$\mathbf{b} = A\mathbf{x} \quad \text{where} \quad A = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{3} & \sqrt{3} & 0 \\ -3 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$





The Singular Value Decomposition: Algebra

$$
\begin{aligned}
A\mathbf{v}_1 &= \sigma_1 \mathbf{u}_1 \\
A\mathbf{v}_2 &= \sigma_2 \mathbf{u}_2 \\
A\mathbf{v}_3 &= \sigma_3 \mathbf{u}_3 \\
\sigma_1 &\geq \sigma_2 > \sigma_3 = 0 \\
\mathbf{u}_1^T \mathbf{u}_1 &= 1 \\
\mathbf{u}_2^T \mathbf{u}_2 &= 1 \\
\mathbf{u}_3^T \mathbf{u}_3 &= 1 \\
\mathbf{u}_1^T \mathbf{u}_2 &= 0 \\
\mathbf{u}_1^T \mathbf{u}_3 &= 0 \\
\mathbf{u}_2^T \mathbf{u}_3 &= 0 \\
\mathbf{v}_1^T \mathbf{v}_1 &= 1 \\
\mathbf{v}_2^T \mathbf{v}_2 &= 1 \\
\mathbf{v}_3^T \mathbf{v}_3 &= 1 \\
\mathbf{v}_1^T \mathbf{v}_2 &= 0 \\
\mathbf{v}_1^T \mathbf{v}_3 &= 0 \\
\mathbf{v}_2^T \mathbf{v}_3 &= 0
\end{aligned}
$$

The Singular Value Decomposition: General

For *any* real $m \times n$ matrix A there exist orthogonal matrices

$$\begin{aligned} U &= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \in \mathcal{R}^{m\times m} \\ V &= \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \in \mathcal{R}^{n\times n} \end{aligned}$$

such that

$$U^T A V = \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p) \in \mathcal{R}^{m\times n}$$

where $p = \min(m, n)$ and $\sigma_1 \geq \ldots \geq \sigma_p \geq 0$. Equivalently,

$$A = U \Sigma V^T \; .$$

- Original formulation: E. Beltrami, 1873
- Stable, efficient algorithm: Golub & Reinsch, 1970

Rank and the Four Subspaces

$$A = U\Sigma V^T = [\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m] \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ 0 & & \cdots & \cdots & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & \cdots & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \\ \mathbf{v}_{r+1} \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

[drawn for $m > n$]

Linear Systems and Reality

$$A\mathbf{x} = \mathbf{b}$$

- $A, \mathbf{b}$ come from measurements $\Rightarrow$ noisy entries
- Systems are typically incompatible
- Reinterpret $A\mathbf{x} = \mathbf{b}$ as $\mathbf{x} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2$
- *Residual vector* $\mathbf{r} = A\mathbf{x} - \mathbf{b}$
- "Least-Squares solution of $A\mathbf{x} = \mathbf{b}$"
- A.k.a. LSE solution (Least Squared-Error)

Incompatibility and Under-Determinacy

- A system can be incompatible *and* its LSE solution can be underdetermined

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 3 \\ x_3 &= 2 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- An LSE solution turns out to be $\mathbf{x} = [1\ 1\ 2]^T$ with residual

$$\mathbf{r} = A\mathbf{x} - \mathbf{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

(split the residual evenly) which has norm $\sqrt{2}$

- Any $\mathbf{x}' = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is as good as $\mathbf{x}$

Uniqueness

- So while the LSE solution always exists, it is not always unique
- It is often convenient to have just one solution, uniquely defined
- Of all solutions, pick the "shortest" one (minimum L_2 norm)
- If you wanted to be cute:

$$\hat{\mathbf{x}} = \arg \min_{\underbrace{\mathbf{x} \in \arg\min_{\mathbf{y}} \|A\mathbf{y} - \mathbf{b}\|}_{\text{a set}}} \|\mathbf{x}\|$$

- $\hat{\mathbf{x}}$ turns out to be unique

The Minimum-Norm LSE Solution

- Theorem: The minimum-norm least-squares solution to a linear system $A\mathbf{x} = \mathbf{b}$, that is, the shortest vector $\mathbf{x}$ that achieves the

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2 ,$$

is unique, and is given by

$$\hat{\mathbf{x}} = V\Sigma^{\dagger}U^T\mathbf{b} \tag{1}$$

where $A = U\Sigma V^T$ is the SVD of A and

$$\Sigma^{\dagger} = \begin{bmatrix} 1/\sigma_1 & & & & & & 0 & \cdots & 0 \\ & \ddots & & & & & & & \\ & & 1/\sigma_r & & & & \vdots & & \vdots \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- The matrix $A^{\dagger} = V\Sigma^{\dagger}U^T$ is called the *pseudoinverse* of A

Homogeneous Linear Systems

- The pseudoinverse yields a fully general LSE solution to $A\mathbf{x} = \mathbf{b}$
- $\hat{\mathbf{x}}$ = the shortest vector $\mathbf{x}$ that achieves the $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|$
- So it works also when $\mathbf{b} = \mathbf{0}$
- However, the solution is trivial:
  The minimum-norm $\mathbf{x}$ that minimizes $\|A\mathbf{x}\|$ is $\mathbf{x} = \mathbf{0}$
- So if this is your problem, you are probably looking at the wrong problem!
- More interesting (and different) problem:

$$\hat{\mathbf{x}} \in \arg \min_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

- A constrained minimization problem: $\mathbf{x} \in$ unit sphere
- Solution is no longer necessarily unique

LSE Solution of the Homogeneous Problem on the Sphere

Let

$$A = U\Sigma V^T$$

be the singular value decomposition of the $m \times n$ matrix A. Then, the last column of V,

$$\mathbf{x} = \mathbf{v}_n$$

is ***a*** unit-norm least-squares solutions to the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}\ .$$

Thus, if $r = \text{rank}(A)$, the *value* of the residual is

$$\min_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \|A\mathbf{v}_n\| = \begin{cases} 0 & \text{if } r < n \\ \sigma_n & \text{otherwise.} \end{cases}$$

In this expression, σ_n is the last singular value of A