Linear Systems

COMPSCI 527 — Computer Vision
Outline

1. Linear Transformations
2. The Solutions of a Linear System
3. Orthogonal Matrices
4. The Singular Value Decomposition
5. The Pseudoinverse
6. Homogeneous Linear System on the Unit Sphere
The Four Fundamental Spaces of a Matrix

\[ \text{null}(A) = \text{span}([0, 0, 1]^T) \]

\[ \text{row space}(A) = \text{span}(\text{first two rows}) \]

\[ \text{range}(A) = \text{span}(\text{first two columns}) \]

\[ \text{left null}(A) = \text{span}(\text{cross product of first two columns}) \]

(comes out to be \(\text{span}([−1, 0, \sqrt{3}]^T)\))

\[ \text{range}(A) \leftrightarrow \text{row space}(A) \]
The Solutions of a Linear System

\[ Ax = b \]

where \( A \) is \( m \times n \), rank \( r \)

- Key point:

  \[ b \notin \text{range}(A) \implies \text{no solutions} \]
  \[ b \in \text{range}(A) \implies \infty^{n-r} \text{ solutions} \]

(An affine space of solutions)
Compatibility

- **Incompatible:**

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
b_3 \\
\end{bmatrix}
\]  
\((b_3 \neq 0)\)

- **Compatible:**

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
0 \\
\end{bmatrix}
\]
Under-Determined System

\[
\begin{bmatrix}
2 & 4 \\
1 & 2 \\
3 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\]
Redundant and Invertible Systems

- Redundant:
  \[
  \begin{bmatrix}
  2 & 1 \\
  1 & 0 \\
  3 & 0 \\
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  \end{bmatrix} =
  \begin{bmatrix}
  4 \\
  1 \\
  3 \\
  \end{bmatrix}
  \]

- Invertible:
  \[
  \begin{bmatrix}
  2 & 1 \\
  3 & 0 \\
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  \end{bmatrix} =
  \begin{bmatrix}
  4 \\
  3 \\
  \end{bmatrix}
  \]

- Inverse:
  \[
  A^{-1} = \frac{1}{3}
  \begin{bmatrix}
  0 & 1 \\
  3 & -2 \\
  \end{bmatrix}
  \]

  \[
  \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3}
  \begin{bmatrix}
  0 & 1 \\
  3 & -2 \\
  \end{bmatrix}
  \begin{bmatrix}
  4 \\
  3 \\
  \end{bmatrix}
  \]

  (This is *not* how linear systems are typically solved)
Summary

- This is not operational!
- Orthogonal matrices $\rightarrow$ SVD $\rightarrow$ rank, bases for the four spaces
- SVD gives us much more
Orthogonal Matrices

- A matrix $V = [v_1, \ldots, v_n]$ is orthogonal if its columns are orthonormal.
- Orthonormal: $v_i^T v_j = \delta_{ij}$ (orthogonal and unit norm).
- Orthogonal matrices have left-inverse $V^T$.
- Square orthogonal matrices have left- and right-inverse $V^T$.
- Orthogonal matrices do not change the norm of vectors:
  $$\|Vx\|^2 = x^T V^T Vx = x^T x = \|x\|^2$$
The Singular Value Decomposition: Geometry

\[ \mathbf{b} = \mathbf{A}\mathbf{x} \quad \text{where} \quad \mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{3} & \sqrt{3} & 0 \\ -3 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \]
The Singular Value Decomposition: Algebra

\[ Av_1 = \sigma_1 u_1 \]
\[ Av_2 = \sigma_2 u_2 \]
\[ Av_3 = \sigma_3 u_3 \]
\[ \sigma_1 \geq \sigma_2 > \sigma_3 = 0 \]
\[ u_1^T u_1 = 1 \]
\[ u_2^T u_2 = 1 \]
\[ u_3^T u_3 = 1 \]
\[ u_1^T u_2 = 0 \]
\[ u_1^T u_3 = 0 \]
\[ u_2^T u_3 = 0 \]
\[ v_1^T v_1 = 1 \]
\[ v_2^T v_2 = 1 \]
\[ v_3^T v_3 = 1 \]
\[ v_1^T v_2 = 0 \]
\[ v_1^T v_3 = 0 \]
\[ v_2^T v_3 = 0 \]
The Singular Value Decomposition: General

For any real $m \times n$ matrix $A$ there exist orthogonal matrices

\[
U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \in \mathbb{R}^{m \times m}
\]
\[
V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}
\]

such that

\[
U^T AV = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p) \in \mathbb{R}^{m \times n}
\]

where $p = \min(m, n)$ and $\sigma_1 \geq \ldots \geq \sigma_p \geq 0$. Equivalently,

\[
A = U\Sigma V^T.
\]

• Original formulation: E. Beltrami, 1873
• Stable, efficient algorithm: Golub & Reinsch, 1970
The Singular Value Decomposition

Rank and the Four Subspaces

\[ A = U \Sigma V^T = [u_1, \ldots, u_r, u_{r+1}, \ldots, u_m] \]

\[ \begin{bmatrix}
\sigma_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_r \\
\vdots & \cdots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
v_1 \\
\vdots \\
v_r \\
v_{r+1} \\
\vdots \\
v_n \\
\end{bmatrix} \]

[drawn for \( m > n \)]
**Linear Systems and Reality**

\[ Ax = b \]

- \( A, b \) come from measurements \( \Rightarrow \) noisy entries
- Systems are typically incompatible
- Reinterpret \( Ax = b \) as \( x \in \arg\min_{x \in \mathbb{R}^n} \| Ax - b \|_2 \)
- *Residual vector* \( r = Ax - b \)
- “Least-Squares solution of \( Ax = b \)”
- A.k.a. LSE solution (Least Squared-Error)
Incompatibility and Under-Determinacy

- A system can be incompatible and its LSE solution can be underdetermined

\[ \begin{align*}
  x_1 + x_2 &= 1 \\
  x_1 + x_2 &= 3 \\
  x_3 &= 2
\end{align*} \]

\[ A = \begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 1 
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
  1 \\
  3 \\
  2 
\end{bmatrix} \]

- An LSE solution turns out to be \( x = [1 \ 1 \ 2]^T \) with residual

\[ r = Ax - b = \begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 1 
\end{bmatrix} \begin{bmatrix}
  1 \\
  3 \\
  2 
\end{bmatrix} - \begin{bmatrix}
  1 \\
  3 \\
  2 
\end{bmatrix} = \begin{bmatrix}
  -1 \\
  -1 \\
  0 
\end{bmatrix}, \]

(split the residual evenly) which has norm \( \sqrt{2} \)

- Any \( x' = \begin{bmatrix}
  1 \\
  1 \\
  2 
\end{bmatrix} + \alpha \begin{bmatrix}
  -1 \\
  1 \\
  0 
\end{bmatrix} \) is as good as \( x \)
Uniqueness

- So while the LSE solution always exists, it is not always unique.
- It is often convenient to have just one solution, uniquely defined.
- Of all solutions, pick the “shortest” one (minimum $L_2$ norm).
- If you wanted to be cute:

$$\hat{x} = \arg\min_{x \in \arg\min_y \|A\,y - b\|} \|x\|$$

- $\hat{x}$ turns out to be unique.
The Minimum-Norm LSE Solution

- Theorem: The minimum-norm least-squares solution to a linear system $A\mathbf{x} = \mathbf{b}$, that is, the shortest vector $\mathbf{x}$ that achieves the

$$ \min_{\mathbf{x}} \| A\mathbf{x} - \mathbf{b} \|_2, $$

is unique, and is given by

$$ \hat{\mathbf{x}} = V\Sigma^\dagger U^T \mathbf{b} \quad (1) $$

where $A = U\Sigma V^T$ is the SVD of $A$ and

$$ \Sigma^\dagger = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1/\sigma_r & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} $$

- The matrix $A^\dagger = V\Sigma^\dagger U^T$ is called the pseudoinverse of $A$
Homogeneous Linear Systems

• The pseudoinverse yields a fully general LSE solution to $A\mathbf{x} = \mathbf{b}$
• $\hat{\mathbf{x}} = \text{the shortest vector } \mathbf{x} \text{ that achieves the } \min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|
• So it works also when $\mathbf{b} = \mathbf{0}$
• However, the solution is trivial:
  The minimum-norm $\mathbf{x}$ that minimizes $\|A\mathbf{x}\|$ is $\mathbf{x} = \mathbf{0}$
• So if this is your problem, you are probably looking at the wrong problem!
• More interesting (and different) problem:

  $\hat{\mathbf{x}} \in \arg \min_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$

• A constrained minimization problem: $\mathbf{x} \in \text{unit sphere}$
• Solution is no longer necessarily unique
LSE Solution of the Homogeneous Problem on the Sphere

Let

\[ A = U \Sigma V^T \]

be the singular value decomposition of the \( m \times n \) matrix \( A \). Then, the last column of \( V \),

\[ x = v_n \]

is a unit-norm least-squares solutions to the homogeneous linear system

\[ Ax = 0 \]

Thus, if \( r = \text{rank}(A) \), the value of the residual is

\[ \min_{\|x\| = 1} \|Ax\| = \|Av_n\| = \begin{cases} 0 & \text{if } r < n \\ \sigma_n & \text{otherwise.} \end{cases} \]

In this expression, \( \sigma_n \) is the last singular value of \( A \).