Ch. 7 - Pushdown Automata

A DFA = (Q, \Sigma, \delta, q_0, F)

\begin{align*}
\text{input tape} & \quad \text{tape head} \\
\begin{array}{cccccccc}
a & a & b & b & a & b & b & \\
0 & 1 & & & & & &
\end{array} \\
\end{align*}

current state

\begin{align*}
0 & \quad 1 \\
\end{align*}
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where

- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) - start stack symbol, \( z \in \Gamma \)
- \( F \subseteq Q \) is set of final states.
- \( \delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of} \ Q \times \Gamma^* \)

Note: the Linz book uses \( z \) for bottom of stack marker, and JFLAP uses \( Z \) (capital Z) for bottom of stack marker.
Example of transitions

\[ \delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\[(q,w,u)\]

Description of a Move:

\[(q_1,aw,bx) \vdash (q_2,w,yx)\]

iff

\[(q_2,y) \in S(q_1,a,b)\]

Definition Let \(M=(Q,\Sigma,\Gamma,\delta,q_0,z,F)\) be a NPDA. \(L(M)=\{w \in \Sigma^* \mid (q_0,w,z) \vdash^*(p,\lambda,u), \ p \in F, \ u \in \Gamma^*\}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.
Example: $L = \{a^n b^n | n \geq 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$

Below illustrates if you push $abcZ$ on the stack, then $a$ is on the top of the stack and $Z$ on the bottom.
Another Definition for Language Acceptance

NPDA M accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* | (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}$$
Example: \( L = \{ a^n b^m c^{n+m} \mid n, m > 0 \} \),
\( \Sigma = \{ a, b, c \}, \ \Gamma = \{ 0, z \} \)
Examples for you to try on your own: (solutions are at the end of the handout).

- $L = \{a^n b^m \mid m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
- $L = \{a^n b^{n+m} c^m \mid n, m > 0\}$, $\Sigma = \{a, b, c\}$
- $L = \{a^n b^{2n} \mid n > 0\}$, $\Sigma = \{a, b\}$
Definition: A PDA 
M=(Q,Σ,Γ,δ,q_0,z,F) is deterministic if for every q ∈ Q, a ∈ Σ ∪ {λ}, b ∈ Γ

1. δ(q, a, b) contains at most 1 element
2. if δ(q, λ, b) ≠ ∅ then δ(q, c, b) = ∅ for all c ∈ Σ

Definition: L is DCFL iff ∃ DPDA M s.t. L=L(M).
Examples:

1. Previous pda for \( \{a^n b^n | n \geq 0\} \) is deterministic? \( \text{yes} \)

2. Previous pda for \( \{a^n b^m c^{n+m} | n, m > 0\} \) is deterministic? \( \text{yes} \)

3. Previous pda for \( \{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\} \) is deterministic? \( \text{non-det} \).
Example: $L = \{a^n b^m | m > n, m, n > 0\}$, 
$\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$

Example: $L = \{a^n b^{n+m} c^m | n, m > 0\}$, 
$\Sigma = \{a, b, c\}$

Example: $L = \{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$