Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
$L = \{ x \mid x \text{ is a positive even integer} \}$

$L$ is closed under

- addition? ✓
- multiplication? ✓
- subtraction? ✗
- division? ✗

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$$L_1 \cup L_2$$
$$L_1 \cap L_2$$
$$L_1L_2$$
$$\overline{L}_1$$
$$L_1^*$$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$\quad L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

Final states in $M$ are nonfinal states in $M'$
nonfinal states in $M$ are final states in $M'$

Note we must start with a DFA
intersection:

$L_1$ and $L_2$ are reg. lang.
⇒ ∃ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = Q \times P$

$\delta':$ d for delta

$d'((q_i, p_j), a) = (q_k, p_l)$ if

$d1(q_i, a) = q_k$ and
$d2(p_j, a) = p_l$

$F' = \{(q_i, p_j) | q_i \text{ in } F_1, \text{ and } p_j \text{ in } F_2\}$
Example:
Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$
Right quotient

Def: \( L_1/L_2 = \{x | xy \in L_1 \text{ for some } y \in L_2\} \)

Example:

\[
L_1 = \{a^*b^* \cup b^*a^*\} \\
L_2 = \{b^n | n \text{ is even, } n > 0\} \\
L_1/L_2 =
\]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

Make $i$ the start state (representing $L_{i}'$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) = \ $

$h(ab^*) = \ $
Questions about regular languages:
L is a regular language.

- Given \( L, \Sigma, w \in \Sigma^* \), is \( w \in L \)?

- Is \( L \) empty?

- Is \( L \) infinite?

- Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \text{ }$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that $L_2 = \{a^n b^n | n > 0 \}$ is ?

• Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \text{ for all } i \geq 0
\]
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  Assume $L$ is regular.
  $\Rightarrow$ $L$ satisfies the pumping lemma.

Choose a long string $w$ in $L$, $|w| \geq m$.

Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.

The pumping lemma does not hold. Contradiction!

$\Rightarrow$ $L$ is not regular. QED.
Example \( L = \{ a^n c b^n | n > 0 \} \)

\( L \) is not regular.

- Proof:
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = \)
Example $L=\{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{ w \in \Sigma^* | n_a(w) > n_b(w) \}$

$L$ is not regular.

• Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

(shown in detail on handout)

$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular. Contradiction!
  
  $L$ is not regular. QED.
Example $L = \{a^3 b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a \quad h(b) = a \quad h(c) = b$
  $h(L) = $
Example L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \}

L is not regular.

- Proof: (proof by contradiction)
  Assume L is regular.
Example: $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.