Which of the following languages are CFL?

- \( L = \{ a^n b^n c^j \mid 0 < n \leq j \} \)
- \( L = \{ a^n b^j a^n b^j \mid n > 0, j > 0 \} \)
- \( L = \{ a^n b^j a^k b^p \mid n + j \leq k + p, n > 0, j > 0, k > 0, p > 0 \} \)
- \( L = \{ a^n b^j a^j b^n \mid n > 0, j > 0 \} \)

**Pumping Lemma for Regular Language’s** Let \( L \) be a regular language, Then there is a constant \( m \) such that \( w \in L, |w| \geq m, w = xyz \) such that

- \( |xy| \leq m \)
- \( |y| \geq 1 \)
- for all \( i \geq 0, xy^iz \in L \)

**Pumping Lemma for CFL’s** Let \( L \) be any infinite CFL. Then there is a constant \( m \) depending only on \( L \), such that for every string \( w \) in \( L \), with \( |w| \geq m \), we may partition \( w = uvxyz \) such that:

\[
|vxy| \leq m, \text{ (limit on size of substring)} \\
|vy| \geq 1, \text{ (} v \text{ and } y \text{ not both empty)} \\
\text{For all } i \geq 0, uv^ixy^iz \in L
\]

**Proof:** (sketch) There is a CFG \( G \) s.t. \( L = L(G) \).
Consider the parse tree of a long string in \( L \).
For any long string, some nonterminal \( N \) must appear twice in the path.
Example: Consider $L = \{a^n b^n c^n : n \geq 1\}$. Show $L$ is not a CFL.

- **Proof:** (by contradiction)
  
  Assume $L$ is a CFL and apply the pumping lemma.
  
  Let $m$ be the constant in the pumping lemma and consider $w = a^m b^m c^m$. Note $|w| \geq m$.
  
  Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i x y^i z \in L$ for $i = 0, 1, 2, \ldots$
  
  Case 1: Neither $v$ nor $y$ can contain 2 or more distinct symbols. If $v$ contains $a$'s and $b$'s, then $uv^2 x y^2 z \notin L$ since there will be $b$'s before $a$'s.
  
  Thus, $v$ and $y$ can be only $a$'s, $b$'s, or $c$'s (not mixed).
  
  Case 2: $v = a^t_1$, then $y = a^t_2$ or $b^t_3$ ($|vxy| \leq m$)
  
  If $y = a^t_2$, then $uv^2 x y^2 z = a^{m+t_1+t_2} b^m c^m \notin L$ since $t_1 + t_2 > 0$, $n(a) > n(b)$'s (number of $a$'s is greater than number of $b$'s)
  
  If $y = b^t_3$, then $uv^2 x y^2 z = a^{m+t_1} b^{m+t_3} c^m \notin L$ since $t_1 + t_3 > 0$, either $n(a) > n(c)$'s or $n(b) > n(c)$'s.
  
  Case 3: $v = b^t_1$, then $y = b^t_2$ or $c^t_3$
  
  If $y = b^t_2$, then $uv^2 x y^2 z = a^m b^{m+t_1+t_2} c^m \notin L$ since $t_1 + t_2 > 0$, $n(b) > n(a)$'s.
  
  If $y = c^t_3$, then $uv^2 x y^2 z = a^m b^{m+t_1} c^{m+t_3} \notin L$ since $t_1 + t_3 > 0$, either $n(b) > n(a)$'s or $n(c) > n(a)$'s.
  
  Case 4: $v = c^t_1$, then $y = c^t_2$
  
  then, $uv^2 x y^2 z = a^m b^{m+c^{m+t_1+t_2}} \notin L$ since $t_1 + t_2 > 0$, $n(c) > n(a)$'s.
  
  Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i x y^i z$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.
Example Why would we want to recognize a language of the type \( \{a^n b^n c^n : n \geq 1\} \)?

Example: Consider \( L = \{a^n b^n c^p : p > n > 0\} \). Show \( L \) is not a CFL.

- **Proof:** Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider \( w = \) ______. Note \(|w| \geq m\).

  Show there is no division of \( w \) into \( uvxwz \) such that \(|vy| \geq 1\), \(|vxy| \leq m\), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

Thus, there is no breakdown of \( w \) into \( uvxwz \) such that \(|vy| \geq 1\), \(|vxy| \leq m\) and for all \( i \geq 0\), \( uv^i xy^i z \) is in \( L \). Contradiction, thus, \( L \) is not a CFL. Q.E.D.
Example: Consider $L = \{a^i b^k : k = j^2\}$. Show $L$ is not a CFL.

- **Proof**: Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider $w = \ldots$

  Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i xy^i z \in L$ for $i = 0, 1, 2, \ldots$

  Case 1: Neither $v$ nor $y$ can contain 2 or more distinct symbols. If $v$ contains $a$’s and $b$’s, then $uv^2 xy^2 z \notin L$ since there will be $b$’s before $a$’s.

  Thus, $v$ and $y$ can be only $a$’s, and $b$’s (not mixed).

  Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i xy^i z \in L$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.

**Exercise**: Prove the following is not a CFL by applying the pumping lemma. (answer is at the end of this handout).

Consider $L = \{a^{2n} b^{2p} c^n d^p : n, p \geq 0\}$. Show $L$ is not a CFL.
Example: Consider $L = \{w\bar{w}w : w \in \Sigma^*\}$, $\Sigma = \{a, b\}$, where $\bar{w}$ is the string $w$ with each occurrence of $a$ replaced by $b$ and each occurrence of $b$ replaced by $a$. For example, $w = baab$, $\bar{w} = abbb$, $\bar{w} = baaabbb$. Show $L$ is not a CFL.

• Proof: Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider $w = \ldots$

Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i xy^i z \in L$ for $i = 0, 1, 2, \ldots$.

Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i xy^i z$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.
**Example:** Consider \( L = \{a^n b^n a^n\} \). \( L \) is a CFL. The pumping lemma should apply!

Let \( m \geq 4 \) be the constant in the pumping lemma. Consider \( w = a^m b^m b^m a^m \).

We can break \( w \) into \( uvxyz \), with:

If you apply the pumping lemma to a CFL, then you should find a partition of \( w \) that works!

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**Chap 8.2 Closure Properties of CFL’s**

**Theorem** CFL’s are closed under union, concatenation, and star-closure.

- **Proof:**
  Given 2 CFG \( G_1 = (V_1, T_1, S_1, P_1) \) and \( G_2 = (V_2, T_2, S_2, P_2) \)

  - **Union:**
    Construct \( G_3 \) s.t. \( L(G_3) = L(G_1) \cup L(G_2) \).
    \( G_3 = (V_3, T_3, S_3, P_3) \)

  - **Concatenation:**
    Construct \( G_3 \) s.t. \( L(G_3) = L(G_1) \circ L(G_2) \).
    \( G_3 = (V_3, T_3, S_3, P_3) \)
- Star-Closure
  Construct $G_3$ s.t. $L(G_3) = L(G_1)^*$
  $G_3 = (V_3, T_3, S_3, P_3)$

QED.

**Theorem** CFL’s are NOT closed under intersection and complementation.

- **Proof:**
  - Intersection:

  - Complementation:
Theorem: CFL’s are closed under regular intersection. If \( L_1 \) is CFL and \( L_2 \) is regular, then \( L_1 \cap L_2 \) is CFL.

- **Proof:** (sketch) This proof is similar to the construction proof in which we showed regular languages are closed under intersection. We take a NPDA for \( L_1 \) and a DFA for \( L_2 \) and construct a NPDA for \( L_1 \cap L_2 \).

\[ M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_0, z, F_1) \] is an NPDA such that \( L(M_1) = L_1 \).

\[ M_2 = (Q_2, \Sigma, \delta_2, q'_0, F_2) \] is a DFA such that \( L(M_2) = L_2 \).

Example of replacing arcs (NOT a Proof!):
Note this is not a proof, but sketches how we will combine the DFA and NPDA. We must formally define $\delta_3$. If

then

Must show

if and only if

Must show:

$w \in L(M_3)$ iff $w \in L(M_1)$ and $w \in L(M_2)$.

QED.
Questions about CFL:

1. Decide if CFL is empty?

2. Decide if CFL is infinite?

Example: Consider \( L = \{a^{2n}b^{2m}c^n d^m : n, m \geq 0 \} \). Show \( L \) is not a CFL.

- **Proof:** Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider \( w = a^{2m}b^{2m}c^m d^m \).

  Show there is no division of \( w \) into \( uvxyz \) such that \( |vy| \geq 1, |vxy| \leq m \), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

  **Case 1:** Neither \( v \) nor \( y \) can contain 2 or more distinct symbols. If \( v \) contains \( a \)'s and \( b \)'s, then \( uv^2xy^2z \notin L \) since there will be \( b \)'s before \( a \)'s.

  Thus, \( v \) and \( y \) can be only \( a \)'s, \( b \)'s, \( c \)'s, or \( d \)'s (not mixed).

  **Case 2:** \( v = a^{t_1} \), then \( y = a^{t_2} \) or \( b^{t_3} \) (\( |vxy| \leq m \))

  If \( y = a^{t_2} \), then \( uv^2xy^2z = a^{2m+t_1+t_2}b^{2m}c^m d^m \notin L \) since \( t_1 + t_2 > 0 \), the number of \( a \)'s is not twice the number of \( c \)'s.

  If \( y = b^{t_3} \), then \( uv^2xy^2z = a^{2m+t_1}b^{2m+t_3}c^m d^m \notin L \) since \( t_1 + t_3 > 0 \), either the number of \( a \)'s (denoted \( n(a) \)) is not twice \( n(c) \) or \( n(b) \) is not twice \( n(d) \).

  **Case 3:** \( v = b^{t_1} \), then \( y = b^{t_2} \) or \( c^{t_3} \)

  If \( y = b^{t_2} \), then \( uv^2xy^2z = a^{2m}b^{2m+t_1+t_2}c^m d^m \notin L \) since \( t_1 + t_2 > 0 \), \( n(b) > 2n(d) \).

  If \( y = c^{t_3} \), then \( uv^2xy^2z = a^{2m}b^{2m+t_1}c^{m+t_3}d^m \notin L \) since \( t_1 + t_3 > 0 \), either \( n(b) > 2n(d) \) or \( 2n(c) > n(a) \).

  **Case 4:** \( v = c^{t_1} \), then \( y = c^{t_2} \) or \( d^{t_3} \)

  If \( y = c^{t_2} \), then \( uv^2xy^2z = a^{2m}b^{2m+t_1+t_2}c^m d^m \notin L \) since \( t_1 + t_2 > 0 \), \( 2n(c) > n(a) \).

  If \( y = d^{t_3} \), then \( uv^2xy^2z = a^{2m}b^{2m+t_1}c^{m+t_3}d^m \notin L \) since \( t_1 + t_3 > 0 \), either \( 2n(c) > n(a) \) or \( 2n(d) > n(b) \).

  **Case 5:** \( v = d^{t_1} \), then \( y = d^{t_2} \)

  then \( uv^2xy^2z = a^{2m}b^{2m}c^m d^{m+t_1+t_2} \notin L \) since \( t_1 + t_2 > 0 \), \( 2n(d) > n(c) \).

  Thus, there is no breakdown of \( w \) into \( uvxyz \) such that \( |vy| \geq 1, |vxy| \leq m \) and for all \( i \geq 0 \), \( uv^i xy^i z \) is in \( L \). Contradiction, thus, \( L \) is not a CFL. Q.E.D.