Methods for Transforming Grammars (Read Ch 6 in Linz Book)

We will consider CFL without $\lambda$. It would be easy to add $\lambda$ to any grammar by adding a new start symbol $S_0$.

$$S_0 \rightarrow S \mid \lambda$$

**Theorem (Substitution)** Let $G$ be a CFG. Suppose $G$ contains

$$A \rightarrow x_1 B x_2$$

where $A$ and $B$ are different variables, and $B$ has the productions

$$B \rightarrow y_1 \mid y_2 \mid \ldots \mid y_n$$

Then can construct $G'$ from $G$ by deleting

$$A \rightarrow x_1 B x_2$$

from $P$ and adding to it

$$A \rightarrow x_1 y_1 x_2 | x_1 y_2 x_2 | \ldots | x_1 y_n x_2$$

Then, $L(G) = L(G')$.

**Example:**

$S \rightarrow aBa$ becomes

$B \rightarrow aS \mid a$

**Definition:** A production of the form $A \rightarrow Ax, A \in V, x \in (V \cup T)^*$ is left recursive.
**Example** Previous expression grammar was left recursive.

\[
\begin{align*}
E & \rightarrow E + T \mid T \\
T & \rightarrow T * F \mid F \\
F & \rightarrow I \mid (E) \\
I & \rightarrow a \mid b
\end{align*}
\]

A top-down parser would want to derive the leftmost terminal as soon as possible. But in the left recursive grammar above, in order to derive a sentential form that has the leftmost terminal, we have to derive a sentential form that has other terminals in it.

Derivation of \( a + b + a + a \) is:

\[
\begin{align*}
E & \Rightarrow E + T \\
E & \Rightarrow E + T + T \\
E & \Rightarrow E + T + T + T \\
T & \Rightarrow T * F \\
T & \Rightarrow T * F + F \\
T & \Rightarrow T * F + F + T + T \\
F & \Rightarrow I \\
F & \Rightarrow I + (E) \\
I & \Rightarrow a \\
I & \Rightarrow a + (E) \\
I & \Rightarrow a + a + (E) \\
I & \Rightarrow a + a + a + (E) \\
I & \Rightarrow a + a + a + a
\end{align*}
\]

We will eliminate the left recursion so that we can derive a sentential form with the leftmost terminal and no other terminals.

**Theorem** (Removing Left recursion) Let \( G = (V, T, S, P) \) be a CFG. Divide productions for variable \( A \) into left-recursive and non left-recursive productions:

\[
\begin{align*}
A & \rightarrow A x_1 \mid A x_2 \mid \ldots \mid A x_n \\
A & \rightarrow y_1 | y_2 | \ldots | y_m
\end{align*}
\]

where \( x_i, y_i \) are in \((V \cup T)^*\).

Then \( G' = (V \cup \{Z\}, T, S, P') \) and \( P' \) replaces rules of form above by

\[
\begin{align*}
A & \rightarrow y_i | y_i Z, \ i = 1, 2, \ldots, m \\
Z & \rightarrow x_i | x_i Z, \ i = 1, 2, \ldots, n
\end{align*}
\]

**Example:**

\[
\begin{align*}
E & \rightarrow E + T \mid T \quad \text{becomes} \\
T & \rightarrow T * F \mid F \quad \text{becomes}
\end{align*}
\]

Now, Derivation of \( a + b + a + a \) is:
Useless productions

S → aB | bA
A → aA
B → Sa
C → cBc | a

What can you say about this grammar?

Theorem (useless productions) Let G be a CFG. Then ∃ G’ that does not contain any useless variables or productions s.t. L(G)=L(G’).

To Remove Useless Productions:

Let G=(V,T,S,P).

I. Compute V₁={Variables that can derive strings of terminals}

1. V₁=∅
2. Repeat until no more variables added
   • For every A∈V with A→x₁x₂...xₙ, xᵢ ∈ (T* ∪ V₁), add A to V₁
3. P₁ = all productions in P with symbols in (V₁ ∪ T)*

Then G₁=(V₁,T,S,P₁) has no variables that can’t derive strings.

II. Draw Variable Dependency Graph

For A → xBy, draw A→B.

Remove productions for V if there is no path from S to V in the dependency graph. Resulting Grammar G’ is s.t. L(G)=L(G’) and G’ has no useless productions.

Example:

S → aB | bA
A → aA
B → Sa | b
C → cBc | a
D → bCb
E → Aa | b
**Theorem** (remove $\lambda$ productions) Let $G$ be a CFG with $\lambda$ not in $L(G)$. Then $\exists$ a CFG $G'$ having no $\lambda$-productions s.t. $L(G) = L(G')$.

**To Remove $\lambda$-productions**

1. Let $V_n = \{A \mid \exists \text{ production } A \rightarrow \lambda \}$
2. Repeat until no more additions
   - if $B \rightarrow A_1A_2\ldots A_m$ and $A_i \in V_n$ for all $i$, then put $B$ in $V_n$
3. Construct $G'$ with productions $P'$ s.t.
   - If $A \rightarrow x_1x_2\ldots x_m \in P$, $m \geq 1$, then put all productions formed when $x_j$ is replaced by $\lambda$ (for all $x_j \in V_n$) s.t. $|\text{rhs}| \geq 1$ into $P'$.

**Example:**

S $\rightarrow$ Ab
A $\rightarrow$ BCB | Aa
B $\rightarrow$ b | $\lambda$
C $\rightarrow$ cC | $\lambda$
Definition Unit Production

\[ A \rightarrow B \]

where \( A, B \in V \).

Consider removing unit productions:

Suppose we have

\[ A \rightarrow B \]
\[ B \rightarrow a | ab \]

But what if we have

\[ A \rightarrow B \]
\[ B \rightarrow C \]
\[ C \rightarrow A \]

Theorem (Remove unit productions) Let \( G=(V,T,S,P) \) be a CFG without \( \lambda \)-productions. Then \( \exists \) CFG \( G'=(V',T',S,P') \) that does not have any unit-productions and \( L(G)=L(G') \).

To Remove Unit Productions:

1. Find for each \( A \), all \( B \) s.t. \( A \rightarrow B \) (Draw a dependency graph)

2. Construct \( G'=(V',T',S,P') \) by
   
   (a) Put all non-unit productions in \( P' \)
   
   (b) For all \( A \rightarrow B \) s.t. \( B \rightarrow y_1 | y_2 | \ldots y_n \in P' \), put \( A \rightarrow y_1 | y_2 | \ldots y_n \in P' \)
Example:

\[ S \rightarrow AB \]
\[ A \rightarrow B \]
\[ B \rightarrow C \mid Bb \]
\[ C \rightarrow A \mid c \mid Da \]
\[ D \rightarrow A \]

**Theorem** Let \( L \) be a CFL that does not contain \( \lambda \). Then \( \exists \) a CFG for \( L \) that does not have any useless productions, \( \lambda \)-productions, or unit-productions.

**Proof**

1. Remove \( \lambda \)-productions
2. Remove unit-productions
3. Remove useless productions

Note order is very important. Removing \( \lambda \)-productions can create unit-productions! QED.
Definition: A CFG is in Chomsky Normal Form (CNF) if all productions are of the form

\[ A \rightarrow BC \quad \text{or} \quad A \rightarrow a \]

where \( A, B, C \in V \) and \( a \in T \).

Theorem: Any CFG \( G \) with \( \lambda \) not in \( L(G) \) has an equivalent grammar in CNF.

Proof:

1. Remove \( \lambda \)-productions, unit productions, and useless productions.

2. For every rhs of length > 1, replace each terminal \( x_i \) by a new variable \( C_j \) and add the production \( C_j \rightarrow x_i \).

3. Replace every rhs of length > 2 by a series of productions, each with rhs of length 2. QED.

Example:

\[
S \rightarrow CBcd \\
B \rightarrow b \\
C \rightarrow Cc \mid e
\]
**Definition:** A CFG is in Greibach normal form (GNF) if all productions have the form

\[ A \rightarrow ax \]

where \( a \in T \) and \( x \in V^* \)

**Theorem** For every CFG \( G \) with \( \lambda \) not in \( L(G) \), \( \exists \) a grammar in GNF.

**Proof:**

1. Rewrite grammar in CNF.
2. Relabel Variables \( A_1, A_2, \ldots A_n \)
3. Eliminate left recursion and use substitution to get all productions into the form:

\[
\begin{align*}
A_i & \rightarrow A_j x_j, \quad j > i \\
Z_i & \rightarrow A_j x_j, \quad j \leq n \\
A_i & \rightarrow a x_i
\end{align*}
\]

where \( a \in T, x_i \in V^* \), and \( Z_i \) are new variables introduced for left recursion.

4. All productions with \( A_n \) are in the correct form, \( A_n \rightarrow ax_n \). Use these productions as substitutions to get \( A_{n-1} \) productions in the correct form. Repeat with \( A_{n-2}, A_{n-3} \), etc until all productions are in the correct form.