Section: Decidability

Computability **A function** f **with domain D is** *computable* **if there exists some TM M such that M computes** f **for all values in its domain.**

Decidability **A problem is** *decidable* **if there exists a TM that can answer yes or no to every statement in the domain of the problem.**

The Halting Problem

Domain: set of all TMs and all strings w.

Question: Given coding of M and w, does M halt on w?

Theorem The halting problem is undecidable.

Proof: (by contradiction)

- **Assume there is a TM H (or algorithm) that solves this problem.**

  TM H has 2 final states, q_y represents yes and q_n represents no.

$$H(w_M, w) = \begin{cases} \text{halts } q_y \text{ if M halts on } w \\ \text{halts } q_n \text{ if M doesn't halt on } w \end{cases}$$

  TM H always halts in a final state.

Construct TM H' from H

$$H'(w_M, w) = \begin{cases} \text{halts} & \text{if M not halt on } w \\ \text{not halt} & \text{if M halts on } w \end{cases}$$

Construct TM $\hat{H}$ from H'

$$\hat{H}(w_M) = \begin{cases} \text{halts} & \text{if M not halt on } w_M \\ \text{not halt} & \text{if M halts on } w_M \end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_{\hat{H}}$.

What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?

Theorem If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

- **Proof: Let L be an RE language over Σ.**

  Let M be the TM such that L=L(M).

  Let H be the TM that solves the halting problem.

A problem A is *reduced* to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.

State-entry problem Given TM M=(Q,$\Sigma, \Gamma, \delta, q_0, B$,F), state $q \in$Q, and string $w \in \Sigma^*$, is state q ever entered when M is applied to w?

This is an undecidable problem!

- **Proof:**

  TM E solves state-entry problem

$$E'(w_M, w) = \begin{cases} \text{M halts on } w & \text{if ?} \\ \text{M doesn't halt on } w & \text{if ?} \end{cases}$$