

Example

$$L = \{a^n b a^n \mid n > 0\}$$

Closure Properties

A set is closed over an operation if

$$\begin{aligned} L_1, L_2 &\in \text{class} \\ L_1 \text{ op } L_2 &= L_3 \\ \Rightarrow L_3 &\in \text{class} \end{aligned}$$

Example

$$L = \{x \mid x \text{ is a positive even integer}\}$$

L is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

$$\begin{aligned} L_1 \cup L_2 \\ L_1 \cap L_2 \\ L_1 L_2 \\ \bar{L}_1 \\ L_1^* \end{aligned}$$

are regular languages.

Proof(sketch)

L_1 and L_2 are regular languages
 $\Rightarrow \exists$ reg. expr. r_1 and r_2 s.t.
 $L_1 = L(r_1)$ and $L_2 = L(r_2)$
 $r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
 \Rightarrow closed under union
 $r_1 r_2$ is r.e. denoting $L_1 L_2$
 \Rightarrow closed under concatenation
 r_1^* is r.e. denoting L_1^*
 \Rightarrow closed under star-closure

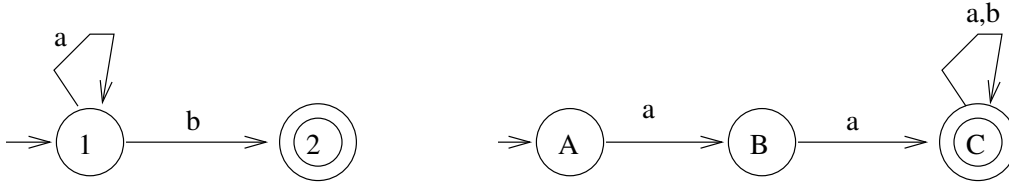
complementation:

L_1 is reg. lang.
 $\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$
Construct M' s.t.

intersection:

L_1 and L_2 are reg. lang.
 $\Rightarrow \exists$ DFA M_1 and M_2 s.t.
 $L_1 = L(M_1)$ and $L_2 = L(M_2)$
 $M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
 $M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
 $Q' =$
 $\delta':$

Example:



Regular languages are closed under

reversal	L^R
difference	$L_1 - L_2$
right quotient	L_1 / L_2
homomorphism	$h(L)$

Right quotient

Def: $L_1 / L_2 = \{x \mid xy \in L_1 \text{ for some } y \in L_2\}$

Example:

$$L_1 = \{a^*b^* \cup b^*a^*\}$$

$$L_2 = \{b^n \mid n \text{ is even, } n > 0\}$$

$$L_1 / L_2 =$$

Theorem If L_1 and L_2 are regular, then L_1 / L_2 is regular.

Proof (sketch)

\exists DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state i do

Make i the start state (representing L'_i)

QED.

Homomorphism

Def. Let Σ, Γ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\begin{aligned} \Sigma &= \{a, b, c\}, \Gamma = \{0, 1\} \\ h(a) &= 11 \\ h(b) &= 00 \\ h(c) &= 0 \end{aligned}$$

$$h(bc) =$$

$$h(ab^*) =$$

Questions about regular languages :

L is a regular language.

- Given $L, \Sigma, w \in \Sigma^*$, is $w \in L$?
- Is L empty?
- Is L infinite?
- Does $L_1 = L_2$?

Ch. 4.3 - Identifying Nonregular Languages

If a language L is finite, is L regular?

If L is infinite, is L regular?

- $L_1 = \{a^n b^m \mid n > 0, m > 0\} =$

- $L_2 = \{a^n b^n \mid n > 0\}$

Prove that $L_2 = \{a^n b^n \mid n > 0\}$ **is ?**

- Proof: Suppose L_2 is regular.
 $\Rightarrow \exists$ DFA M that recognizes L_2

Pumping Lemma: Let L be an infinite regular language. \exists a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$\begin{aligned} |xy| &\leq m \\ |y| &\geq 1 \\ xy^iz &\in L \quad \text{for all } i \geq 0 \end{aligned}$$

Meaning: Every long string in L (the constant m above corresponds to the finite number of states in M in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in L .

To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.

Assume L is regular.

$\Rightarrow L$ satisfies the pumping lemma.

Choose a long string w in L , $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).

Show that there is NO division of w into xyz (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \forall i \geq 0$.

The pumping lemma does not hold. Contradiction!

$\Rightarrow L$ is not regular. QED.

Example $L = \{a^n cb^n | n > 0\}$

L is not regular.

- **Proof:**

Assume L is regular.

\Rightarrow the pumping lemma holds.

Choose $w =$ _____ where m is the constant in the pumping lemma. (Note that w must be chosen such that $|w| \geq m$.)

The only way to partition w into three parts, $w = xyz$, is such that x contains 0 or more a 's, y contains 1 or more a 's, and z contains 0 or more a 's concatenated with cb^m . This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

It should be true that $xy^iz \in L$ for all $i \geq 0$.

Example $L = \{a^n b^{n+s} c^s \mid n, s > 0\}$

L is not regular.

• **Proof:**

Assume L is regular.

\Rightarrow the pumping lemma holds.

Choose $w =$

The only way to partition w into three parts, $w = xyz$, is such that x contains 0 or more a 's, y contains 1 or more a 's, and z contains 0 or more a 's concatenated with the rest of the string $b^{m+s} c^s$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

Example $\Sigma = \{a, b\}$, $L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

• **Proof:**

Assume L is regular.

\Rightarrow the pumping lemma holds.

Choose $w =$

So the partition is:

Example $L = \{a^3b^nc^{m-3} \mid n > 3\}$

L is not regular.

• **Proof:**

Assume L is regular. \Rightarrow the pumping lemma holds.

Choose $w = a^3b^m c^{m-3}$ where m is the constant in the pumping lemma. There are three ways to partition w into three parts, $w = xyz$. 1) y contains only a 's 2) y contains only b 's and 3) y contains a 's and b 's

We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide w into three parts s.t. the pumping lemma constraints were true).

Case 1: (y contains only a 's). Then x contains 0 to 2 a 's, y contains 1 to 3 a 's, and z contains 0 to 2 a 's concatenated with the rest of the string $b^m c^{m-3}$, such that there are exactly 3 a 's. So the partition is:

$$x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}$$

where $k \geq 0$, $j > 0$, and $k + j \leq 3$ for some constants k and j .

It should be true that $xy^i z \in L$ for all $i \geq 0$.

$xy^2 z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-k-j} b^m c^{m-3}) = a^{3+2j} b^m c^{m-3} \notin L$ since $j > 0$, there are too many a 's. Contradiction!

Case 2: (y contains only b 's) Then x contains 3 a 's followed by 0 or more b 's, y contains 1 to $m - 3$ b 's, and z contains 3 to $m - 3$ b 's concatenated with the rest of the string c^{m-3} . So the partition is:

$$x = a^3 b^k \quad y = b^j \quad z = b^{m-k-j} c^{m-3}$$

where $k \geq 0$, $j > 0$, and $k + j \leq m - 3$ for some constants k and j .

It should be true that $xy^i z \in L$ for all $i \geq 0$.

$xy^0 z = a^3 b^{m-j} c^{m-3} \notin L$ since $j > 0$, there are too few b 's. Contradiction!

Case 3: (y contains a 's and b 's) Then x contains 0 to 2 a 's, y contains 1 to 3 a 's, and 1 to $m - 3$ b 's, z contains 3 to $m - 1$ b 's concatenated with the rest of the string c^{m-3} . So the partition is:

$$x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j} c^{m-3}$$

where $3 \geq k > 0$, and $m - 3 \geq j > 0$ for some constants k and j .

It should be true that $xy^i z \in L$ for all $i \geq 0$.

$xy^2 z = a^3 b^j a^k b^m c^{m-3} \notin L$ since $j, k > 0$, there are b 's before a 's. Contradiction!

\Rightarrow There is no partition of w .

$\Rightarrow L$ is not regular!. QED.

To Use Closure Properties to prove L is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**

Assume L is regular.

Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

closure properties \Rightarrow L' is regular.

Contradiction!

L is not regular. QED.

Example $L = \{a^3b^nc^{n-3} \mid n > 3\}$

L is not regular.

- **Proof:** (proof by contradiction)

Assume L is regular.

Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$

$$h(a) = a \quad h(b) = a \quad h(c) = b$$

$$h(L) =$$

Example $L = \{a^n b^m a^m \mid m \geq 0, n \geq 0\}$

L is not regular.

- **Proof:** (proof by contradiction)
Assume L is regular.

Example: $L_1 = \{a^n b^n a^n \mid n > 0\}$

L_1 is not regular.

- **Proof:**

Assume L_1 is regular.

Goal is to try to construct $\{a^n b^n \mid n > 0\}$ which we know is not regular.

Let $L_2 = \{a^*\}$. L_2 is regular.

By closure under right quotient, $L_3 = L_1 \setminus L_2 = \{a^n b^n a^p \mid 0 \leq p \leq n, n > 0\}$ is regular.

By closure under intersection, $L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n \mid n > 0\}$ is regular.

Contradiction, already proved L_4 is not regular!

Thus, L_1 is not regular. QED.