Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
$L = \{ x \mid x \text{ is a positive even integer} \}$

$L$ is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1L_2$
$\overline{L}_1$
$L_1^*$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' =$

$\delta'$:
Example:
Regular languages are closed under

reversal \( L^R \)
difference \( L_1 - L_2 \)
right quotient \( L_1 / L_2 \)
homomorphism \( h(L) \)
Right quotient

Def: $L_1/L_2 = \{x \mid xy \in L_1 \text{ for some } y \in L_2\}$

Example:

$L_1 = \{a^*b^* \cup b^*a^*\}$
$L_2 = \{b^n \mid n \text{ is even, } n > 0\}$
$L_1/L_2 =$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L_{i}'$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

$$h(a) = 11$$
$$h(b) = 00$$
$$h(c) = 0$$

$$h(bc) =$$

$$h(ab^*) =$$
Questions about regular languages:

L is a regular language.

• Given L, \( \Sigma \), \( w \in \Sigma^* \), is \( w \in L \)?

• Is \( L \) empty?

• Is \( L \) infinite?

• Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language \( L \) is finite, is \( L \) regular?

If \( L \) is infinite, is \( L \) regular?

- \( L_1 = \{a^n b^m | n > 0, m > 0\} = \)
- \( L_2 = \{a^n b^n | n > 0\} \)
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

• Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
\begin{align*}
|xy| & \leq m \\
|y| & \geq 1 \\
xy^iz & \in L \text{ for all } i \geq 0
\end{align*}
\]
To Use the Pumping Lemma to prove $L$ is not regular:

- Proof by Contradiction.
  
  Assume $L$ is regular.
  
  $\Rightarrow$ $L$ satisfies the pumping lemma.
  
  Choose a long string $w$ in $L$, $|w| \geq m$.
  
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^i z \in L \ \forall \ i \geq 0$.
  
  The pumping lemma does not hold. Contradiction!
  
  $\Rightarrow$ $L$ is not regular. QED.
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
Example $L=\{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w = \ldots$

  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* | n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular. Contradiction!

  $L$ is not regular. QED.
Example \( L = \{a^3b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.

- Proof: (proof by contradiction)
  - Assume \( L \) is regular.
  - Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
    - \( h(a) = a \) \( h(b) = a \) \( h(c) = b \)
    - \( h(L) = \)
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof: (proof by contradiction)**
  Assume $L$ is regular.
Example: \( L_1 = \{a^n b^n a^n | n > 0\} \)

\( L_1 \) is not regular.