Constraint Satisfaction Problems
CompSci 370

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With thanks to Kris Hauser for some slides

Checking for Solution Existence

- In some problems, we don't care about a path, but about a configuration that has a *desired property*
- Instead of a goal, we have a target, which can be a set of states that satisfy some property

- We call the set of properties that legal solutions must obey constraints
- We call these problems constraint satisfaction problems (CSPs)

CSP Examples

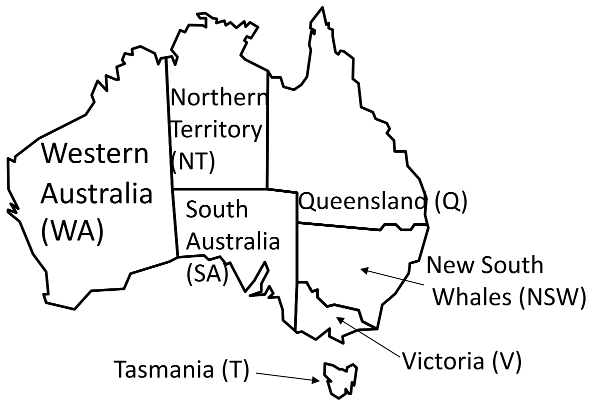
- Satisfying curriculum/major requirements
- Sudoku
- Seating arrangements at a party
- LSAT Questions:
  http://www.thelsattrainer.com/sample-lsat-logic-games.html

CSPs

- Specifying CSPs
- One view: Search with special goal criteria
- CSP definition (general):
  - Variables $X_1,\ldots,X_n$
  - Variable X_i has domain D_i
  - Constraints $C_1,\ldots,C_m$
  - Solution: Each variable gets a value from its domain such that no constraints violated
- CSP examples…
  - http://www.csplib.org/

CSP Example

Graph coloring:



Problem: Assign Red, Green and Blue so that no 2 adjacent regions have the same color. (3-coloring)

CSP as a Search Problem

- *n* variables $X_1, \ldots, X_n$
- Valid assignment: $\{X_{i1} \leftarrow v_{i1}, \ldots, X_{ik} \leftarrow v_{ik}\}$, $0 \leq k \leq n$, such that the values $v_{i1}, \ldots, v_{ik}$ satisfy all constraints relating the variables $X_{i1}, \ldots, X_{ik}$
- Complete assignment: one where $k = n$
  [if all variable domains have size d, there are $O(d^n)$ complete assignments]
- **States**: valid assignments
- **Initial state**: empty assignment {}, i.e. k = 0
- **Successor** of a state:

$\{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}\} \rightarrow \{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}, X_{ik+1}\leftarrow v_{ik+1}\}$

- **Goal test**: k = n

Backtracking Search

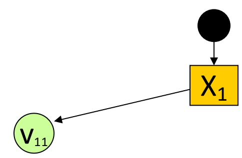
- Essentially a simplified depth-first algorithm using recursion

Backtracking Search
(3 variables)

●

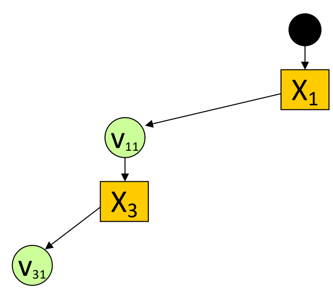
Assignment = {}

Backtracking Search
(3 variables)



Assignment = {(X_1,v_{11})}

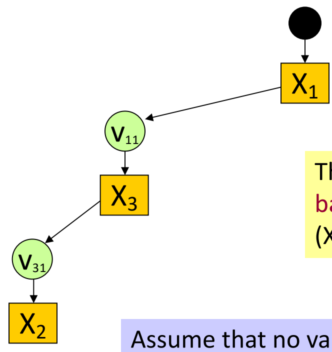
Backtracking Search
(3 variables)



Assignment = {(X_1,v_{11}), (X_3,v_{31})}

Backtracking Search

(3 variables)



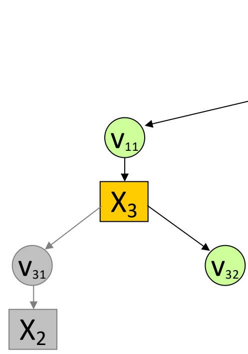
Then, the search algorithm backtracks to the previous variable (X_3) and tries another value

Assume that no value of X_2 leads to a valid assignment

Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search

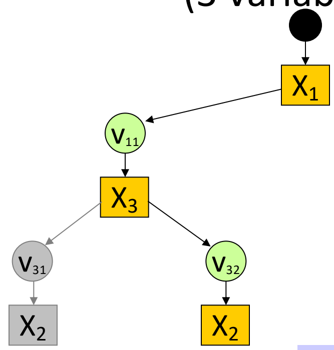
(3 variables)



Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

Backtracking Search

(3 variables)



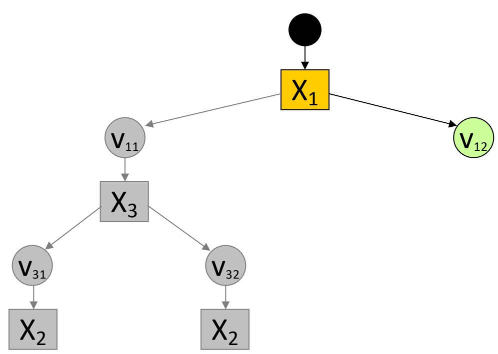
The search algorithm backtracks to the previous variable (X_3) and tries another value. But assume that X_3 has only two possible values. The algorithm backtracks to X_1

Assume again that no value of X_2 leads to a valid assignment

Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

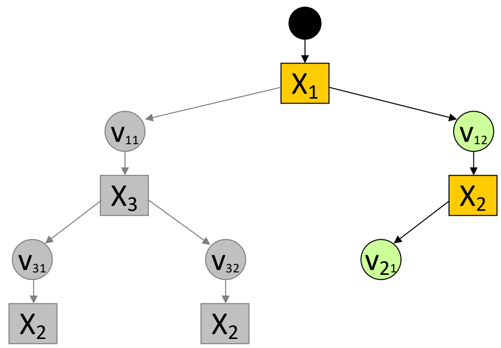
Backtracking Search

(3 variables)



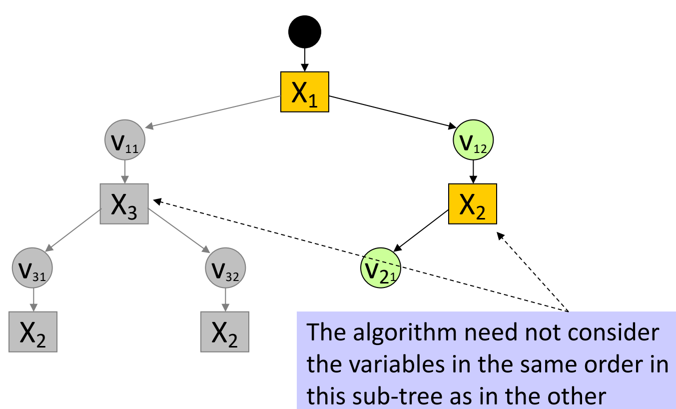
Assignment = $\{(X_1,v_{12})\}$

Backtracking Search
(3 variables)



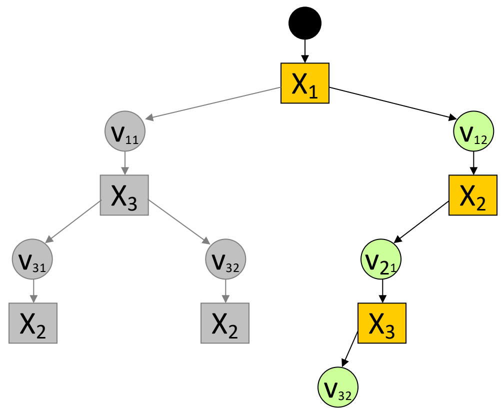
Assignment = {(X_1,v_{12}), (X_2,v_{21})}

Backtracking Search
(3 variables)



Assignment = {(X_1,v_{12}), (X_2,v_{21})}

Backtracking Search
(3 variables)



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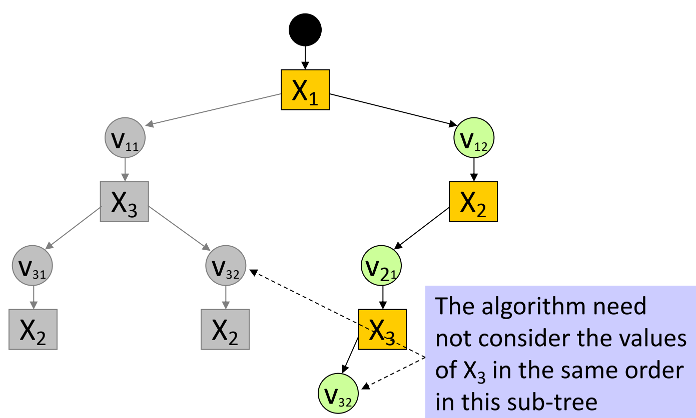
Backtracking Search
(3 variables)



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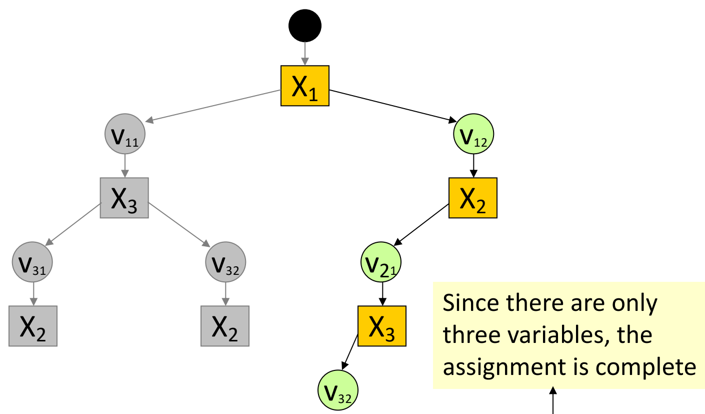
Backtracking Search

(3 variables)



Assignment = $\{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}$

Backtracking Algorithm

CSP-BACKTRACKING(A)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. If A is valid then
      i. result ← CSP-BACKTRACKING(A)
      ii. If result ≠ failure then return result
   c. Remove (X←v) from A
5. Return failure

Efficiency of CSP-Backtracking

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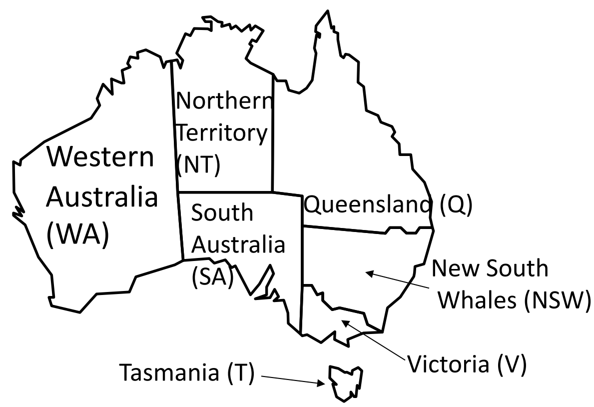
Practical Efficiency of CSP Algorithms

- Fundamental trade off
  - Time spent ruling out bad/impossible choices
  - Time spent searching
- Try to find the sweet spot where you quickly rule out bad/unpromising choices
- Compare with sweet spot for heuristics in A*

CSP Example Revisited

Graph coloring:



Problem: Assign Red, Green and Blue so that no 2 adjacent regions have the same color. (3-coloring)

Example Contd.

- Variables: {WA, NT, Q, SA, NSW, V, T}
- Domains: {R,G,B}
- Constraints:

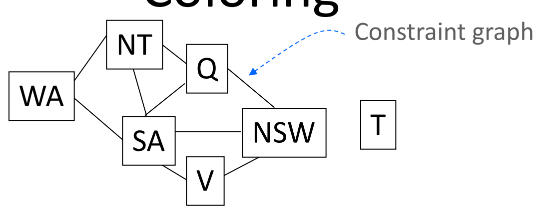
  For WA – NT:{(R,G), (R,B), (G,B), (G,R), (B,R), (B,G)}
- We have a table for each adjacent pair
- Note: Many possible ways to express constraints

Forward Checking

- Idea: Assignments to variables immediately rule out certain assignments to other variables
- Remove illegal/invalid options from the domains other variables
- You probably do this when you play Sudoku!

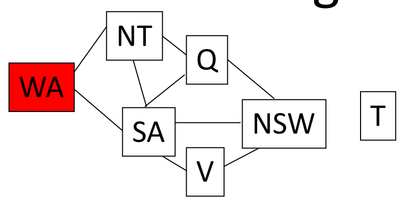
5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

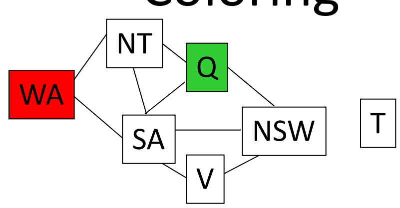
Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB

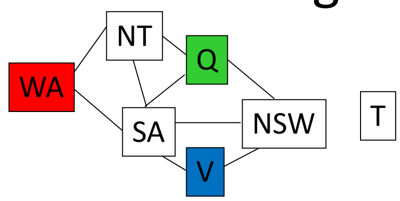
Forward checking removes the value Red of NT and of SA

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking in Map Coloring

Empty set: the current assignment
{(WA ← R), (Q ← G), (V ← B)}
does not lead to a solution

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking (General Form)

Whenever a pair (X←v) is added to assignment A do:

For each variable Y not in A do:

For every constraint C relating Y to the variables in A do:

Remove all values from Y's domain that do not satisfy C

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
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No need any more to verify that A is valid

Modified Backtracking Algorithm

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Need to pass down the updated variable domains

Modified Backtracking Algorithm

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   c. If no variable has an empty domain then
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      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

1) Which variable X_i should be assigned a value next?
   → Most-constrained-variable heuristic
   → Most-constraining-variable heuristic

2) In which order should its values be assigned?
   → Least-constraining-value heuristic

NOTE: Different use of the word "heuristic" from A*
Don't confuse these two! You will only get questions about heuristics as functions from states to reals!

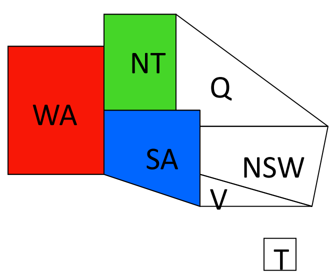
Most-Constrained-Variable Heuristic

1) Which variable X_i should be assigned a value next?

Select the variable with the smallest remaining domain

[Rationale: Minimize the branching factor]

Map Coloring



- SA's remaining domain has size 1 (value B remaining)
- Q's remaining domain has size 2
- NSW's, V's, and T's remaining domains have size 3

→ Select SA

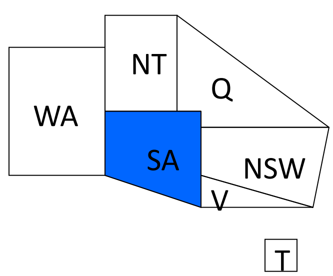
Most-Constraining-Variable Heuristic

1) Which variable X_i should be assigned a value next?

   Among the variables with the smallest remaining domains (ties with respect to the most-constrained-variable heuristic), select the one that appears in the largest number of constraints on variables not in the current assignment

   [Rationale: Increase future elimination of values, to reduce future branching factors]

Map Coloring



- Before any value has been assigned, all variables have a domain of size 3, but SA is involved in more constraints (5) than any other variable

→ Select SA and assign a value to it (e.g., Blue)

Least-Constraining-Value Heuristic

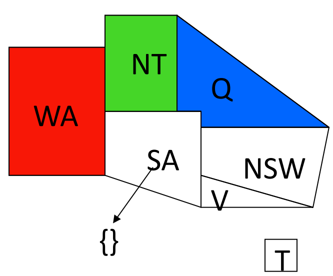
2) In which order should X's values be assigned?

Select the value of X that removes the smallest number of values from the domains of those variables which are not in the current assignment

[Rationale: Since only one value will eventually be assigned to X, pick the least-constraining value first, since it is the most likely not to lead to an invalid assignment]

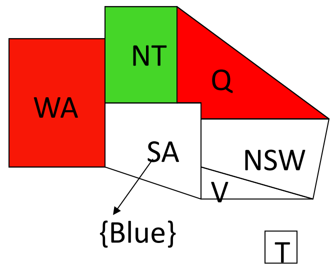
[Note: Using this heuristic requires performing a forward-checking step for *every value*, not just for the selected value]

Map Coloring



- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 values for SA, while assigning Red would leave 1 value

Map Coloring



- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

→ So, assign Red to Q

More Advanced Constraint Propagation

- Forward checking can't discover all possible consequences that could lead to failure
- (Doing this in general would require solving the entire problem, so we shouldn't expect a free lunch here.)
- AC3 (see textbook) is an advanced algorithm that is a good trade off between efficiency and effectiveness
- But how hard are CSPs, really?

Digression: NP-Hardness

- NP hardness is not an AI topic
- You will not be tested on it explicitly, but

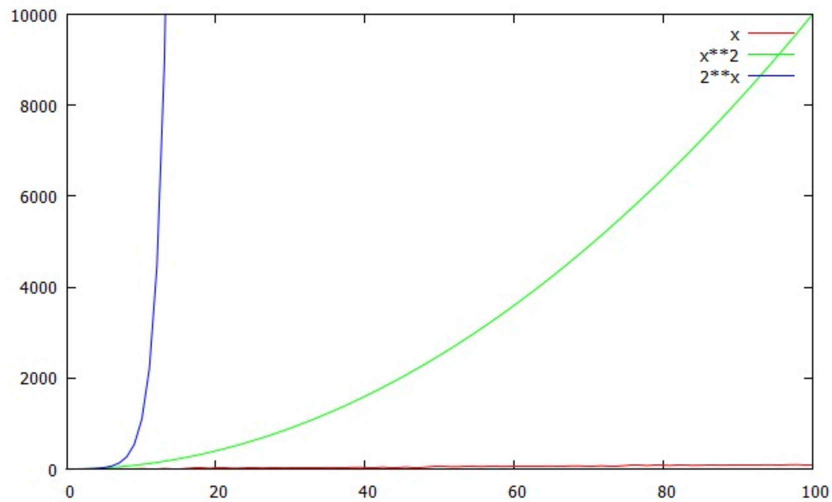
- It's important for all computer scientists
- Understanding it will deepen your understanding of AI (and other CS) topics
- You will be expected to understand its relevance and use for AI problems

- Eat your vegetables; they're good for you

P and NP

- P and NP are about decision problems
- P is set of problems that can be solved in polynomial time
- NP is a superset of P
- NP is the set of problems that:
  - Have solutions which can be verified in polynomial time or, equivalently,
  - can be solved by a non-deterministic Turing machine in polynomial time (OK if you don't know what that means yet)
- Roughly speaking:
  - Problems in P are tractable – can be solved in a reasonable amount of time, and Moore's law helps
  - Some problems in NP *might* not be tractable

Scaling

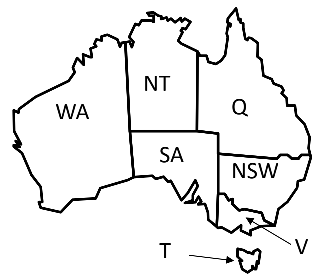


Isn't P big?

- P includes $O(n)$, $O(n^2)$, $O(n^{10})$, $O(n^{100})$, etc.
- Clearly $O(n^{10})$ isn't something to be excited about – not practical
- Computer scientists are very clever at making things that are in P efficient
- First algorithms for some problems are often quite expensive, e.g., $O(n^3)$, but research often brings this down

Understanding the class NP

- A class of *decision problems* (Yes/No)
- Solutions can be verified in polynomial time
- Examples:
  - Graph coloring:



  - Sortedness: [1 2 3 4 5 8 7]

NP-hardness

- Many problems in AI are NP-hard (or worse)
- What does this mean?
- NP-hard = as hard as hardest problems in NP
- Identifying a problem as NP hard means:
  - You probably shouldn't waste time trying to find a polynomial time solution
  - If you find a polynomial time solution, either
    - You have a bug
    - Find a place on your shelf for your Turing award
- NP hardness is a major triumph (and failure) for computer science theory

NP-hardness

- Why it is a failure:
  - There is a huge class of problems with no known efficient solutions
  - We have failed, as a community, to either find efficient solutions or prove that none exist

- Why it is a triumph:
  - We have a developed a precise language for talking about these problems
  - We have developed sophisticated ways to reason about and categorize the problems we don't know how to solve efficiently

P=NP?

- ***Biggest open question in CS***

- Can NP-hard problems be solved in poly time?
- Probably not, but nobody has been able to prove it yet

- Many false starts, e.g.:

  http://www.nytimes.com/2009/10/08/science/Wpolynom.html

How challenging is "P=NP?"

- Princeton University CS department
- See: http://www.cs.princeton.edu/general/bricks.php
- Photo from: http://stuckinthebubble.blogspot.com/2009/07/three-interesting-points-on-princeton.html

Hardness of CSPs

- CSPs are known to be NP-hard
  (for most reasonable formulations of the problem)
- Bad news: Don't bother trying to find a general, efficient way to solve CSPs
- Good news: Many problems can be solved much faster than the worst (exponential) case in practice
- So-so news: Sometimes you just need to run a solver and see what happens
  - You might get an answer quickly
  - You might just wait, and wait, and wait…

CSP Conclusions

- CSPs are a general language for describe a large family of problems
- Might require exponential time (worst case)

- Advanced algorithms exist that try to discover bad choices quickly, reducing the search space
  - Microsoft Solver Foundation
  - CPLEX