Due date: March 3, 2022

Problem 1: [15 pts] Let a circle $C \in \mathbb{R}^2$ have center $c$ and radius $r$. The power $\text{pow}(x, C)$ of a point $x \in \mathbb{R}^2$ with respect to circle $C$ is defined as:

$$\text{pow}(x, C) := \|x - c\|^2 - r^2.$$  

The power diagram of a finite set of circles $S$ is a decomposition of $\mathbb{R}^2$ into regions such that all points in the same region have the same nearest neighbor under the power distance. That is, a cell for a given $s \in S$ is defined as $\text{Pow}(s) := \{x \in \mathbb{R}^2 \mid \text{pow}(x, s) \leq \text{pow}(x, t) \forall t \in S\}$.

(i) Describe the bisector of two circles in the power diagram.

(ii) Prove that the the power diagram has linear complexity.

(iii) Describe an $O(n \log n)$ algorithm to compute the power diagram of a set of circles $S$.

Problem 2: [10 pts] Let $S$ be a set of $n$ points in $\mathbb{R}^2$, such that each point $p \in S$ has a positive weight $w(p)$. We define the weighted distance of an arbitrary point $x \in \mathbb{R}^2$ from $p$ to be $d(x, p) := \|x - p\|w(p)$. The weighted Voronoi cell of a point $p \in S$ is as usual defined as $\text{Vor}(p) := \{x \in \mathbb{R}^2 \mid d(x, p) \leq d(x, p') \forall p' \in S\}$. The resulting diagram is known as the multiplicative weighted Voronoi diagram.

(i) Describe the bisector of two points in the multiplicative weighted Voronoi diagram.

(ii) What is the complexity of the multiplicative weighted Voronoi diagram? Justify your answer.

Problem 3: [15 pts] A $k$-clustering of a set $P$ of $n$ points in the plane is a partitioning of $P$ into $p$ non-empty subsets $P_1, \ldots, P_k$. Define the distance $d$ between any pair $P_i, P_j$ of clusters to be the minimum distance between one point from $P_i$ and one point from $P_j$, that is,

$$d(P_i, P_j) := \min_{p \in P_i, q \in P_j} \|p - q\|.$$  

We want to find a $k$-clustering (for given $k$ and $P$) that maximizes the minimum distance between clusters.

(i) Suppose the minimum distance between clusters is achieved by points $p \in P_i$ and $q \in P_j$. Prove that $pq$ is an edge of the Delaunay triangulation of $P$.

(ii) Give an $O(n \log n)$ time algorithm to compute a $k$-clustering maximizing the minimum distance between clusters. (Hint: Use a Union-Find data structure.)

Problem 4: [10 pts] Let $S$ be a set of $n$ points in $\mathbb{R}^d$, and let $k$ be an integer with $1 \leq k \leq \binom{n}{2}$. Suppose we order the distances between pairs of points of $S$ in non-decreasing order. We want to compute the pair of points that realize the $k^{th}$ smallest distance, $\delta$. For a parameter $\delta > 0$, a pair $(x, y)$ of $S$ is an $\varepsilon$-approximate $k^{th}$ closest pair if $(1 - \varepsilon)\delta \leq \|x - y\| \leq (1 + \varepsilon)\delta$. Show that an $\varepsilon$-approximate $k^{th}$ closest pair can be computed in $O(n\varepsilon^{-d} \log n)$ time. (Hint: Compute a WSPD of $S$.)