Due date: March 3, 2022

Problem 1: [15 pts] Let a circle $C \in \mathbb{R}^2$ have center c and radius r. The power $pow(x, C)$ of a point $x \in \mathbb{R}^2$ with respect to circle C is defined as:

$$pow(x, C) := \|x - c\|^2 - r^2.$$

The *power diagram* of a finite set of circles S is a decomposition of $\mathbb{R}^2$ into regions such that all points in the same region have the same nearest neighbor under the power distance. That is, a cell for a given $s \in S$ is defined as $\text{Pow}(s) := \{x \in \mathbb{R}^2 \mid pow(x, s) \leq pow(x, t) \forall t \in S\}$.

(i) Describe the bisector of two circles in the power diagram.

(ii) Prove that the the power diagram has linear complexity.

(iii) Descibe an $O(n \log n)$ algorithm to compute the power diagram of a set of circles S.

Problem 2: [10 pts] Let S be a set of n points in $\mathbb{R}^2$, such that each point $p \in S$ has a positive weight $w(p)$. We define the weighted distance of an arbitrary point $x \in \mathbb{R}^2$ from p to be $d(x, p) := \|x - p\| w(p)$. The weighted Voronoi cell of a point $p \in S$ is as usual defined as $\text{Vor}(p) := \{x \in \mathbb{R}^2 \mid d(x, p) \leq d(x, p') \forall p' \in S\}$. The resulting diagram is known as the *multiplicative weighted Voronoi diagram*.

(i) Describe the bisector of two points in the multiplicative weighted Voronoi diagram.

(ii) What is the complexity of the multiplicative weighted Voronoi diagram? Justify your answer.

Problem 3: [15 pts] A k-clustering of a set P of n points in the plane is a partitioning of P into p non-empty subsets $P_1, \ldots, P_k$. Define the distance d between any pair P_i, P_j of clusters to be the minimum distance between one point from P_i and one point from P_j, that is,

$$d(P_i, P_j) := \min_{p \in P_i,\ q \in P_j} \|p - q\|.$$

We want to find a k-clustering (for given k and P) that maximizes the minimum distance between clusters.

(i) Suppose the mimimum distance between clusters is achieved by points $p \in P_i$ and $q \in P_j$. Prove that pq is an edge of the Delaunay triangulation of P.

(ii) Give an $O(n \log n)$ time algorithm to compute a k-clustering maximizing the minimum distance between clusters. (**Hint:** *Use a Union-Find data structure.*)

Problem 4: [10 pts] Let S be a set of n points in $\mathbb{R}^d$, and let k be an integer with $1 \leq k \leq \binom{n}{2}$. Suppose we order the distances between pairs of points of S in non-decreasing order. We want to compute the pair of points that realize the k^{th} smallest distance, δ. For a parameter $\varepsilon > 0$, a pair (x, y) of S is an ε-approximate k^{th} closest pair if $(1 - \varepsilon)\delta \leq \|x - y\| \leq (1 + \varepsilon)\delta$. Show that an ε-approximate k^{th} closest pair can be computed in $O(n\varepsilon^{-d} \log n)$ time. (**Hint:** *Compute a WSPD of S.*)