Section: Properties of Regular Languages

Example

\[ L = \{a^n b a^n \mid n > 0\} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \; \text{op} \; L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\( L = \{ x \mid x \text{ is a positive even integer} \} \)

\( L \) is closed under

- addition? \( \text{yes} \)
- multiplication? \( \text{yes} \)
- subtraction? \( \text{no} \)
- division? \( \text{no} \)

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[
L_1 \cup L_2 \\
L_1 \cap L_2 \\
L_1 L_2 \\
\bar{L}_1 \\
L_1^*
\]

are regular languages.
Proof(sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$\quad L_1 = L(r_1)$ and $L_2 = L(r_2)$
$\quad r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union

$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

final states in $M$ are nonfinal states in $M'$

nonfinal states in $M$ are final states in $M'$
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$s'(((q_i, p_j), a)) = (q_k, p_l)$ iff

$s_1(q_i, a) = q_k \in M_1$ and

$s_2(p_j, a) = p_l \in M_2$

$F' = \{ (q_i, p_j) \in Q' | q_i \in F_1 \text{ and } p_j \in F_2 \}$
Example:
Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$
Right quotient

Def: $L_1/L_2 = \{x \mid xy \in L_1 \text{ for some } y \in L_2\}$

Example:

$L_1 = \{a^*b^* \cup b^*a^*\}$
$L_2 = \{b^n \mid n \text{ is even, } n > 0\}$
$L_1/L_2 = \exists \alpha \ast b^* \exists$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

For each state $i$ do

    Make $i$ the start state (representing $L_i'$)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

$$h(a) = 11$$

$$h(b) = 00$$

$$h(c) = 0$$

$$h(bc) =$$

$$h(ab^*) =$$
Questions about regular languages:

L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?

• Is L empty?

• Is L infinite?

• Does L₁ = L₂?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \emptyset$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

• Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

- $|xy| \leq m$
- $|y| \geq 1$
- $xy^i z \in L$ for all $i \geq 0$
To Use the Pumping Lemma to prove L is not regular:

- Proof by Contradiction.
  Assume L is regular.
  \( \Rightarrow \) L satisfies the pumping lemma.
  Choose a long string \( w \) in L, \( |w| \geq m \).
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \( |xy| \leq m, |y| \geq 1 \) and \( xy^iz \in L \ \forall \ i \geq 0 \).
  The pumping lemma does not hold. Contradiction!
  \( \Rightarrow \) L is not regular. QED.
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  - Assume $L$ is regular.
  - $\Rightarrow$ the pumping lemma holds.
  - Choose $w =$
Example $L=\{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{ w \in \Sigma^* \mid n_a(w) > n_b(w) \}$

$L$ is not regular.

• Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w = \ldots$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$
(shown in detail on handout)
$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular.
  
  Contradiction!

  $L$ is not regular. QED.
Example $L = \{a^3b^nc^{n-3} | n > 3\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a \quad h(b) = a \quad h(c) = b$
  $h(L) = \_\_\_\_\_\_\_\_\_$
Example \( L = \{a^n b^m a^m | m \geq 0, n \geq 0\} \)

\( L \) is not regular.

- **Proof: (proof by contradiction)**
  Assume \( L \) is regular.
Example: \( L_1 = \{a^n b^n a^n | n > 0\} \)

\( L_1 \) is not regular.