Methods for Transforming Grammars (Read Ch 6 in Linz Book)

We will consider CFL without $\lambda$. It would be easy to add $\lambda$ to any grammar by adding a new start symbol $S_0$,

\[ S_0 \rightarrow S \mid \lambda \]

**Theorem (Substitution)** Let $G$ be a CFG. Suppose $G$ contains

\[ A \rightarrow x_1 B x_2 \]

where $A$ and $B$ are different variables, and $B$ has the productions

\[ B \rightarrow y_1 \mid y_2 \mid \ldots \mid y_n \]

Then can construct $G'$ from $G$ by deleting

\[ A \rightarrow x_1 B x_2 \]

from $P$ and adding to it

\[ A \rightarrow x_1 y_1 x_2 \mid x_1 y_2 x_2 \mid \ldots \mid x_1 y_n x_2 \]

Then, $L(G)=L(G')$.

**Example:**

$S \rightarrow aBa$ becomes

$B \rightarrow aS \mid a$  

**Definition:** A production of the form $A \rightarrow Ax$, $A \in V$, $x \in (V \cup T)^*$ is *left recursive*. 

**Example** Previous expression grammar was left recursive.

\[
\begin{align*}
E & \rightarrow E + T \mid T \\
T & \rightarrow T * F \mid F \\
F & \rightarrow I \mid (E) \\
I & \rightarrow a \mid b
\end{align*}
\]

A top-down parser would want to derive the leftmost terminal as soon as possible. But in the left recursive grammar above, in order to derive a sentential form that has the leftmost terminal, we have to derive a sentential form that has other terminals in it.

Derivation of \(a+b+a+a\) is:

\[
E \Rightarrow E + T \Rightarrow E + T + T \Rightarrow E + T + T + T \Rightarrow a + T + T + T
\]

We will eliminate the left recursion so that we can derive a sentential form with the leftmost terminal and no other terminals.

**Theorem** (Removing Left recursion) Let \(G=(V,T,S,P)\) be a CFG. Divide productions for variable \(A\) into left-recursive and non left-recursive productions:

\[
\begin{align*}
A & \rightarrow A x_1 \mid A x_2 \mid \ldots \mid A x_n \\
A & \rightarrow y_1 \mid y_2 \mid \ldots \mid y_m
\end{align*}
\]

where \(x_i, y_i\) are in \((V \cup T)^*\).

Then \(G'=(V \cup \{Z\}, T, S, P')\) and \(P'\) replaces rules of form above by

\[
\begin{align*}
A & \rightarrow y_i | y_i Z, \ i=1,2,\ldots, m \\
Z & \rightarrow x_i | x_i Z, i=1,2,\ldots, n
\end{align*}
\]

**Example:**

\[
\begin{align*}
E & \rightarrow E + T | T \quad \text{becomes} \\
T & \rightarrow T * F | F \quad \text{becomes}
\end{align*}
\]

Now, Derivation of \(a+b+a+a\) is:
Useless productions

S → aB | bA
A → aA
B → Sa
C → cBc | a

What can you say about this grammar?

Theorem (useless productions) Let G be a CFG. Then ∃ G’ that does not contain any useless variables or productions s.t. L(G)=L(G’).

To Remove Useless Productions:

Let G=(V,T,S,P).

I. Compute V₁={Variables that can derive strings of terminals}

1. V₁=∅
2. Repeat until no more variables added
   • For every A∈V with A→x₁x₂…xₙ, xᵢ∈(T* ∪ V₁), add A to V₁
3. P₁ = all productions in P with symbols in (V₁ ∪ T)*

Then G₁=(V₁,T,S,P₁) has no variables that can’t derive strings.

II. Draw Variable Dependency Graph

For A → xBy, draw A→B.

Remove productions for V if there is no path from S to V in the dependency graph. Resulting Grammar G’ is s.t. L(G)=L(G’) and G’ has no useless productions.

Example:

S → aB | bA
A → aA
B → Sa | b
C → cBc | a
D → bCb
E → Aa | b
Theorem (remove λ productions) Let G be a CFG with λ not in L(G). Then ∃ a CFG G’ having no λ-productions s.t. L(G)=L(G’).

To Remove λ-productions

1. Let $V_n = \{ A \mid \exists \text{ production } A \rightarrow \lambda \}$

2. Repeat until no more additions
   - if $B \rightarrow A_1 A_2 \ldots A_m$ and $A_i \in V_n$ for all $i$, then put B in $V_n$

3. Construct G’ with productions P’ s.t.
   - If $A \rightarrow x_1 x_2 \ldots x_m \in P$, $m \geq 1$, then put all productions formed when $x_j$ is replaced by $\lambda$ (for all $x_j \in V_n$) s.t. $|\text{rhs}| \geq 1$ into P’.

Example:

S $\rightarrow$ Ab  
A $\rightarrow$ BCB | Aa  
B $\rightarrow$ b | $\lambda$  
C $\rightarrow$ cC | $\lambda$
**Definition** Unit Production

\[ A \rightarrow B \]

where \( A, B \in V \).

**Consider removing unit productions:**

Suppose we have

\[
\begin{align*}
A &\rightarrow B \\
B &\rightarrow a | ab
\end{align*}
\]

But what if we have

\[
\begin{align*}
A &\rightarrow B \\
B &\rightarrow C \\
C &\rightarrow A
\end{align*}
\]

**Theorem** (Remove unit productions) Let \( G=(V,T,S,P) \) be a CFG without \( \lambda \)-productions. Then \( \exists \) CFG \( G'=(V',T',S,P') \) that does not have any unit-productions and \( L(G)=L(G') \).

**To Remove Unit Productions:**

1. Find for each \( A, \) all \( B \) s.t. \( A \Rightarrow B \) (Draw a dependency graph)
2. Construct \( G'=(V',T',S,P') \) by
   (a) Put all non-unit productions in \( P' \)
   (b) For all \( A \Rightarrow B \) s.t. \( B \Rightarrow y_1|y_2|\ldots|y_n \in P' \), put \( A \Rightarrow y_1|y_2|\ldots|y_n \in P' \)
Theorem Let $L$ be a CFL that does not contain $\lambda$. Then $\exists$ a CFG for $L$ that does not have any useless productions, $\lambda$-productions, or unit-productions.

Proof

1. Remove $\lambda$-productions
2. Remove unit-productions
3. Remove useless productions

Note order is very important. Removing $\lambda$-productions can create unit-productions! QED.
**Definition:** A CFG is in Chomsky Normal Form (CNF) if all productions are of the form

\[ A \rightarrow BC \quad \text{or} \quad A \rightarrow a \]

where \( A, B, C \in V \) and \( a \in T \).

**Theorem:** Any CFG \( G \) with \( \lambda \) not in \( L(G) \) has an equivalent grammar in CNF.

**Proof:**

1. Remove \( \lambda \)-productions, unit productions, and useless productions.
2. For every rhs of length \( > 1 \), replace each terminal \( x_i \) by a new variable \( C_j \) and add the production \( C_j \rightarrow x_i \).
3. Replace every rhs of length \( > 2 \) by a series of productions, each with rhs of length 2. QED.

**Example:**

\[
\begin{align*}
S & \rightarrow CBcd \\
B & \rightarrow b \\
C & \rightarrow Cc \mid e
\end{align*}
\]
**Definition:** A CFG is in Greibach normal form (GNF) if all productions have the form

\[ A \rightarrow ax \]

where \( a \in T \) and \( x \in V^* \)

**Theorem** For every CFG G with \( \lambda \) not in \( L(G) \), \( \exists \) a grammar in GNF.

**Proof:**

1. Rewrite grammar in CNF.
2. Relabel Variables \( A_1, A_2, \ldots, A_n \)
3. Eliminate left recursion and use substitution to get all productions into the form:

\[
\begin{align*}
A_i & \rightarrow A_j x_j, \quad j > i \\
Z_i & \rightarrow A_j x_j, \quad j \leq n \\
A_i & \rightarrow ax_i
\end{align*}
\]

where \( a \in T \), \( x_i \in V^* \), and \( Z_i \) are new variables introduced for left recursion.

4. All productions with \( A_n \) are in the correct form, \( A_n \rightarrow ax_n \). Use these productions as substitutions to get \( A_{n-1} \) productions in the correct form. Repeat with \( A_{n-2}, A_{n-3}, \) etc until all productions are in the correct form.